The relations (A. 4) imply that

$$\frac{d\mu_i^s}{d\tau} + \frac{d\mu_i^{s-}}{d\tau}\bigg|_{t_k-\tau} = -\frac{\partial H^*}{\partial x_i^s} - \frac{\partial H^*}{\partial x_i^{s-}}\bigg|_{t_k-\tau}$$

$$\frac{d\mu_i^s}{d\tau}\bigg|_{t_k-\tau} + \frac{d\mu_i^{s-}}{d\tau} = -\frac{\partial H^*}{\partial x_i^s}\bigg|_{t_k-\tau} - \frac{\partial H^*}{\partial x_i^{s-}}$$

By writing out the right sides, cancelling identical terms and using (A. 6) and (A. 7) and the obvious relations

$$\frac{\partial \varphi_j^{\nu-}}{\partial x_i^{s-}}\bigg|_{t_k-\tau} = \frac{\partial \varphi_j^{\nu}}{\partial x_i^{s}}, \qquad \frac{\partial \varphi_j^{\nu}}{\partial x_i^{s-}}\bigg|_{t_k-\tau} = \frac{\partial \varphi_j^{\nu-}}{\partial x_i^{s}}$$

we find that the left sides of (A. 6) also satisfy the system (1. 9).

The theorem has been proved in its entirety.

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BOUNDARY LAYER AND ITS INTERACTION WITH THE INTERIOR STATE OF STRESS OF AN ELASTIC THIN SHELL

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The theory of the interior state of stress constructed in [1] in conformity with the scheme described in [2] is supplemented by an asymptotic boundary layer theory (a theory of edge states of stress) and the question of boundary layer interaction with the interior state of stress is solved for a thin elastic isotropic shell.

A two-dimensional linear theory of thin elastic shells is formulated at the end. It is based on the results herein and in [1], and is an extension of the classical theory of shells in the sense that it permits a more exact construction of the interior state of stress and, in a certain approximation, the investigation of edge elastic phenomena not taken into account by classical theory. The interior state of stress is computed by the method proposed by using equations and boundary conditions of classical theory, which are insignificantly modified, and the computations of the edge stresses reduces to the construction of a linear combination of solutions of certain auxiliary plane and antiplane problems with standard conditions independent of the geometric properties of the shell and of the nature of its loading.

1. Let us utilize a tri-orthogonal $(\alpha_1, \alpha_2, \alpha_3)$ coordinate system specifying a point by the radius-vector $\mathbf{P}$ according to formula

$$\mathbf{P} = \mathbf{M}(\alpha_1, \alpha_2) + \alpha_3 \mathbf{n} \tag{1.1}$$

where $\mathbf{M}(\alpha_1, \alpha_2,)$ is the radius-vector of some surface referred to the lines of curvature, and $\mathbf{n}$ is the unit normal vector to this surface.

We call the surfaces $\alpha_3 = 0$, $\alpha_3 = \pm h$ the middle and face surfaces of the shell, respectively. The constant h is understood to be half the shell thickness.

Let σ_{ij} and v_i denote the stresses and displacements in the chosen coordinate system, and let us examine the two following groups of quantities:

$$S_{11} = (1 + \alpha_3 / R_2)\sigma_{11}, \quad S_{13} = (1 + \alpha_3 / R_2)\sigma_{13}, \quad S_{22} = (1 + \alpha_3 / R_1)\sigma_{22}$$
$$S_{33} = (1 + \alpha_3 / R_1)(1 + \alpha_3 / R_2)\sigma_{33}, \quad U_1 = h^{-1}v_1, \quad U_3 = h^{-1}v_3 \tag{1.2}$$
$$T_{12} = (1 + \alpha_3 / R_1)\sigma_{12}, \quad T_{21} = (1 + \alpha_3 / R_2)\sigma_{21}, \quad T_{23} = (1 + \alpha_3 / R_1)\sigma_{23}$$
$$V_2 = h^{-1}v_2 \tag{1.3}$$

These formulas replace the symmetric stresses σ_{ij} by the nonsymmetric stresses S_{ij} T_{pq}, and the true displacements v_i by the dimensionless displacements U_1, V_2, U_3. Moreover, the stresses and dimensionless displacements are separated into two groups, which will later be denoted by S and T, respectively

$$S = (S_{11}, S_{22}, S_{33}, S_{13}, U_1, U_3), \qquad T = (T_{12}, T_{21}, T_{23}, V_2) \tag{1.4}$$

2. Let us consider the differential equations of elasticity theory for an isotropic body. In the arbitrary tri-orthogonal coordinate system under the assumption of no mass forces, they can be written thus:

equilibrium equations

$$\frac{\partial}{\partial \alpha_i}(H_j H_k \sigma_{ii}) + \frac{\partial}{\partial \alpha_j}(H_i H_k \sigma_{ij}) + \frac{\partial}{\partial \alpha_k}(H_i H_j \sigma_{ik}) -$$
$$- H_k \frac{\partial H_j}{\partial \alpha_i}\sigma_{jj} - H_j \frac{\partial H_k}{\partial \alpha_i}\sigma_{kk} + H_k \frac{\partial H_i}{\partial \alpha_j}\sigma_{ji} + H_j \frac{\partial H_i}{\partial \alpha_k}\sigma_{ki} = 0$$
$$(i \neq j \neq k = 1, 2, 3)$$

Hooke's law

$$E\left(\frac{1}{H_i}\frac{\partial v_i}{\partial \alpha_i} + \frac{1}{H_i H_j}\frac{\partial H_i}{\partial \alpha_j}v_j + \frac{1}{H_i H_k}\frac{\partial H_i}{\partial \alpha_k}v_k\right) = \sigma_{ii} - \nu(\sigma_{jj} + \sigma_{kk})$$
$$E\left[\frac{H_i}{H_j}\frac{\partial}{\partial \alpha_j}\left(\frac{v_i}{H_i}\right) + \frac{H_j}{H_i}\frac{\partial}{\partial \alpha_i}\left(\frac{v_j}{H_j}\right)\right] = 2(1 + \nu)\sigma_{ij}$$
$$(i \neq j \neq k = 1, 2, 3)$$

Replacing here σ_{ij} and v_i by S, T by means of (1.2), (1.3) and considering the coordinates (1.1) to have been selected, we obtain a system of equations which can be written briefly as

$$N_n(S, T) = 0 \quad (n = 1, \ldots, 6), \qquad M_m(S, T) = 0 \quad (m = 1, 2, 3, 4) \tag{2.1}$$

The following formulas elucidate the notation introduced here:

$$N_1(S,T) = \partial_1 S_{11} + \partial_2 T_{12} + (1 + \alpha_3 / R_1)\partial_3 S_{13} + k_2(S_{11} - S_{22}) +$$
$$+ k_1(T_{12} + T_{21}) + 2S_{13} / R_1$$
$$M_1(S,T) = \partial_1 T_{21} + \partial_2 S_{22} + (1 + \alpha_3 / R_2)\partial_3 T_{23} + k_1(S_{22} - S_{11}) +$$
$$+ k_2(T_{12} + T_{21}) + 2T_{23} / R_2$$

$$N_2(S,T) = \partial_1 S_{31} + \partial_2 T_{32} + \partial_3 S_{33} + k_2 S_{13} + k_1 T_{23} - S_{11} / R_1 - S_{22} / R_2$$

$$N_3(S,T) = E(1 + \alpha_3 / R_2)(\partial_1 U_1 + k_1 V_2 + U_3 / R_1) - \\ -h^{-1}[(1 + \alpha_3 / R_1) S_{11} - \nu(1 + \alpha_3 / R_2) S_{22} - \nu S_{33}]$$

$$N_4(S,T) = E(1 + \alpha_3/R_1)(\partial_2 V_2 + k_2 U_1 + U_3 / R_2) - h^{-1}[(1 + \alpha_3/R_2) S_{22} - \\ - \nu(1 + \alpha_3 / R_1) S_{11} - \nu S_{33}]$$

$$N_5(S,T) = E(1 + \alpha_3 / R_1)(1 + \alpha_3 / R_2)\partial_3 U_3 - \\ -h^{-1}[S_{33} - \nu(1 + \alpha_3 / R_1) S_{11} - \nu(1 + \alpha_3 / R_2) S_{22}]$$

$$M_2(S,T) = E[(1 + \alpha_3/R_1)\partial_2 U_1 + (1 + \alpha_3/R_2)\partial_1 V_2 - k_1(1 + \alpha_3 / R_2) U_1 - \\ - k_2(1 + \alpha_3 / R_1) V_2] - 2h^{-1}(1 + \nu)(1 + \alpha_3 / R_1) T_{21} \qquad (2.2)$$

$$M_3(S,T) = E[(1 + \alpha_3 / R_2)\partial_1 V_2 + (1 + \alpha_3 / R_1)\partial_2 U_1 - k_2(1 + \alpha_3 / R_1) V_2 - \\ - k_1(1 + \alpha_3 / R_2) U_1] - 2h^{-1}(1 + \nu)(1 + \alpha_3 / R_2) T_{12}$$

$$N_6(S,T) = E(1 + \alpha_3 / R_2)[(1 + \alpha_3 / R_1)\partial_3 U_1 - U_1 / R_1 + \partial_1 U_3] - \\ - 2h^{-1}(1 + \nu)(1 + \alpha_3 / R_1) S_{13}$$

$$M_4(S,T) = E(1 + \alpha_3 / R_1)[(1 + \alpha_3 / R_2)\partial_3 V_2 - V_2 / R_2 + \partial_2 U_3] - \\ -2h^{-1}(1 + \nu)(1 + \alpha_3 / R_2) T_{23}$$

where

$$\partial_i = \frac{1}{A_i}\frac{\partial}{\partial \alpha_i} \quad (i = 1, 2); \qquad \partial_3 = \frac{\partial}{\partial \alpha_3}, \quad k_j = \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i} \quad (i \neq j = 1, 2) \qquad (2.3)$$

The equations $M_2 = 0$ and $M_3 = 0$ in the system (2.1), (2.2) are duplicates, and this can be seen by examining the quantities T_{12} and T_{21} by means of (1.3).

3. The boundary layer is understood to be that stress-strain state which is localized near some normal section of the shell (formed by the normals of the middle surface), and damps with distance from it.

Let us assume that this normal section is given by the equation $\alpha_1 = 0$. (The domain of applicability of the proposed theory is thereby restricted to the assumption that the boundary layer originates near the lines of curvature). To be definite, let us assume that the parameters (α_1, α_2) have the dimensionality of a length, and are selected so that A_1, A_2 are commensurate with unity, i.e. that the arclengths of lines on the surface are commensurate with the corresponding increments in the parameters (α_1, α_2). It is sufficient to satisfy this latter condition just near $\alpha_1 = 0$, where the boundary layer is generated, and this new constraint on the generality is not essential.

Let us introduce a change in independent variables by means of the formulas

$$\alpha_1 = R\varkappa^{-q}\xi_1, \qquad \alpha_2 = R\varkappa^{-p}\xi_2, \qquad \alpha_3 = R\varkappa^{-q}\zeta \qquad (3.1)$$

where R is the characteristic radius of curvature of the middle surface, $\varkappa$ a large dimensionless parameter, and p and q are integers which are chosen so that the following equalities are satisfied

$$\varkappa^{-q} = h_* = h / R, \qquad \varkappa^{-p} = h_*^{\,t}, \qquad t = p / q \qquad (3.2)$$

It is henceforth considered that R is commensurate with unity in the selected length scale, and that differentiation with respect to ξ_1, ξ_2, ζ does not change the order of the desired quantities. This latter is equivalent to the assumption that the desired quantities are magnified, respectively, by $\varkappa^q$, $\varkappa^p$, $\varkappa^q$ times when differentiated with respect to α_1, α_2, α_3.

This means, firstly, that the number t in (3.1), (3.2) agrees in its meaning with the index of variability of the desired state of stress in α_2 (in particular, along the edge line $\alpha_1 = 0$), and secondly, that the indices of variability along the normal to the middle surface, and the tangential normal to $\alpha_1 = 0$ equal one.

The number p, and the variability of the boundary layer in α_2 in addition, are defined by the conditions of the problem: it must be selected so that t would equal the index of variability of those self-equilibrated edge effects which generate the boundary layer. The equality of the indices of variability in α_1 and α_3 to one will as yet be an arbitrary assumption. It is later justified by the fact that it results in a consistent iteration process.

Let us execute the change of variables (3.1) in (2.1), (2.2), and let us expand the coefficients of these equations in a Taylor series near $\alpha_1 = 0$, i.e. let us set

$$P = \sum_{\rho=0}^{\infty} \alpha_1{}^\rho P_\rho = \sum_{\rho=0}^{\infty} R^\rho \varkappa^{-q\rho} \xi_1{}^\rho P_\rho$$

where P can be understood to be any coefficient in (2.2), for example

$$k_i = k_{i0} + R\varkappa^{-q}\xi_1 k_{i1} + \ldots, \qquad \frac{1}{R_i} = \frac{1}{R_{i0}} + R\varkappa^{-q}\xi_1 \left(\frac{1}{R_i}\right)_1 + \ldots$$

The symbols ∂_i and ∂_3 defined by the first two equations in (2.3), are hence transformed thus:

$$\partial_1 = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1} = \frac{1}{A_1}\frac{\varkappa^q}{R}\frac{\partial}{\partial\xi_1} = \frac{\varkappa^q}{R}\sum_{\rho=0}^{\infty}\varkappa^{-q\rho}\xi_1{}^\rho d_{1\rho} \tag{3.3}$$

$$\partial_2 = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2} = \frac{1}{A_2}\frac{\varkappa^p}{R}\frac{\partial}{\partial\xi_2} = \frac{\varkappa^p}{R}\sum_{\rho=0}^{\infty}\varkappa^{-q\rho}\xi_1{}^\rho d_{2\rho}, \qquad \partial_3 = \frac{\varkappa^q}{R} d_3$$

Here

$$d_{1\rho} = R^\rho \left(\frac{1}{A_1}\right)_\rho \frac{\partial}{\partial\xi_1}, \qquad d_{2\rho} = R^\rho \left(\frac{1}{A_2}\right)_\rho \frac{\partial}{\partial\xi_2}, \quad d_3 = \frac{\partial}{\partial\zeta} \tag{3.4}$$

$(1/A_i)_\rho$ are coefficients of the Taylor series expansion of $1/A_i$ in α_1

Let us insert these results into (2.1), (2.2), let us replace the α_3 which enters explicitly, by (3.1), and h by (3.2), and let us expand (2.2) in powers of the small parameter $\varkappa^{-q}$, by considering $\varkappa^{-p}$ as an independent parameter. We obtain

$$\sum_{s=0}^{\infty} \varkappa^{-qs}\left[\frac{\varkappa^q}{R} N_{ns}^{13}(S,T) + \frac{\varkappa^p}{R} N_{ns}^{2}(S,T) + N_{ns}^{*}(S,T)\right] = 0 \quad (n=1,\ldots,6)$$

$$\sum_{s=0}^{\infty} \varkappa^{-qs}\left[\frac{\varkappa^q}{R} M_{ms}^{13}(S,T) + \frac{\varkappa^p}{R} M_{ms}^{2}(S,T) + M_{ms}^{*}(S,T)\right] = 0 \quad (m=1,2,3,4) \tag{3.5}$$

The superscripts in (3.5) mean that the following component groupings were taken in (2.2):

$M_m{}^{13}, N_n{}^{13}$ are sets of components containing either the differentiation symbols ∂_1 or ∂_3, or the factor h^{-1};

$M_m{}^2$, $N_n{}^2$ are sets of components containing the differentiation symbol ∂_2;

$M_m{}^*$, $N_m{}^*$ are sets of components not in the two preceding groups.

The common factor $\varkappa^q/R$, which is written down explicitly in (3.5), is extracted outside the brackets in the expressions for $M_m{}^{13}$, $N_n{}^{13}$, and the rest of these expressions is expanded in powers of $\varkappa^{-q}$. The other terms in (3.5) are analogously constructed.

Let us present the expansions of some of the operators in (3.5)

$$N_{10}^{13} = d_{10}S_{11} + d_3S_{13}, \qquad N_{20}^{13} = d_{10}S_{13} + d_3S_{33}$$
$$N_{30}^{13} = Ed_{10}U_1 - [S_{11} - \nu(S_{22} + S_{33})], \qquad N_{40}^{13} = -[S_{22} - \nu(S_{11} + S_{33})] \tag{3.6}$$
$$N_{50}^{13} = Ed_3U_3 - [S_{33} - \nu(S_{11} + S_{22})], \qquad N_{60}^{13} = E(d_3U_1 + d_{10}U_3) - 2(1+\nu)S_{13}$$

$$M_{10}^{13} = d_{10}T_{21} + d_3T_{23}, \qquad M_{20}^{13} = E\,d_{10}V_2 - 2(1+\nu)T_{21}$$
$$M_{30}^{13} = E\,d_{10}V_2 - 2(1+\nu)T_{12}, \qquad M_{40}^{13} = E\,d_3V_2 - 2(1+\nu)T_{23} \tag{3.7}$$

$$N_{10}^{2} = d_{20}T_{12}, \quad N_{20}^{2} = d_{20}T_{23}, \quad N_{40}^{2} = Ed_{20}V_2, \quad N_{30}^{2} = N_{50}^{2} = N_{60}^{2} = 0 \tag{3.8}$$

$$M_{10}^{2} = d_{20}S_{22}, \quad M_{20}^{2} = Ed_{20}U_1, \quad M_{30}^{2} = Ed_{20}U_1, \quad M_{40}^{2} = Ed_{20}U_3 \tag{3.9}$$

$$N_{11}^{13} = \xi_1 d_{11}S_{11} + \zeta(R/R_{10})d_3S_{13}, \qquad N_{21}^{13} = \xi_1 d_{11}S_{13}$$
$$M_{11}^{13} = \xi_1 d_{11}T_{21} + \zeta(R/R_{20})\,d_3T_{23}$$
$$N_{10}{}^* = k_{20}(S_{11} - S_{22}) + k_{10}(T_{12} + T_{21}) + 2S_{13}/R_{10} \tag{3.10}$$
$$M_{10}{}^* = k_{10}(S_{22} - S_{11}) + k_{20}(T_{12} + T_{21}) + 2T_{23}/R_{20}$$
$$N_{20}{}^* = k_{20}S_{13} + k_{10}T_{23} - S_{11}/R_{10} - S_{22}/R_{20}$$

4. Let us give T and S in (3.5) as expansions in descending powers of $\varkappa$

$$T_a = \sum_{t=0}^{\infty} \varkappa^{-t}T_a^{(t)}, \qquad S_a = \varkappa^{-q+p}\sum_{t=0}^{\infty} \varkappa^{-t}S_a^{(t)} \tag{4.1}$$

(the value of the subscript a is explained below).

Let us require that the coefficients of identical powers of $\varkappa$, starting with the highest, vanish.

Since all the operators N and M are linear, then

$$N(S_a, T_a) = \varkappa^{-q+p}\sum_{t=0}^{\infty} \varkappa^{-t}N(S_a^{(t)}, 0) + \sum_{t=0}^{\infty} \varkappa^{-t}N(0, T_a^{(t)})$$

and analogously for M. Making note of this, we obtain a sequence of systems of equations

$$\sum_{s=0}^{\infty} N_{ns}^{13}(S_a^{(\tau-q+p-sq)}, 0) + \sum_{s=0}^{\infty} N_{ns}^{2}(0, T_a^{(\tau-q+p-sq)}) +$$
$$+ R\sum_{s=0}^{\infty} N_{ns}{}^*(0, T_a^{(\tau-q-sq)}) + \left\{R\sum_{s=0}^{\infty} N_{ns}{}^*(S_a^{(\tau-2q+p-sq)}, 0)\right\} = 0$$

$$\sum_{s=0}^{\infty} M_{ms}^{13}(0, T_a^{(\tau-sq)}) + \left\{\sum_{s=0}^{\infty} M_{ms}^{2}(S_a^{(\tau-2q+2p-sq)}, 0)\right\} + \tag{4.2}$$
$$+ R\sum_{s=0}^{\infty} M_{ms}^{*}(0, T_a^{(\tau-q-sq)}) + \left\{R\sum_{s=0}^{\infty} M_{ms}^{*}(S_a^{(\tau-2q+p-sq)}, 0)\right\} = 0$$

Here and henceforth it is considered everywhere that

$$T^{(r)} = S^{(r)} \equiv 0 \quad (r<0) \tag{4.3}$$

and moreover, it is taken into account that by virtue of (2.2)

$$N_{ns}^{13}(0, T) \equiv 0, \quad M_{ms}^{13}(S, 0) \equiv 0, \quad N_{ns}^{2}(S, 0) \equiv 0, \quad M_{ms}^{2}(0, T) \equiv 0$$

If only the first r members for T_a and $r - q + p$ members for S_a are taken into account in the sums (4.1), i.e. if for terms having factors of at least $\varkappa^{-r}$ are discarded, we shall say that (S_a, T_a) are constructed with a formal asymptotic error of the order of $O(\varkappa^{-r})$. This concept has also been used in [1], where a formal asymptotic error of the order of

$$\varepsilon = O(\varkappa^{-2q+2p}) = O(h_*^{2-2t}) \tag{4.4}$$

was assumed herein for the theory of the interior state of stress.

We shall also construct a boundary layer theory with the formal asymptotic error (4.4). Then, in the exponents for T_a in (4.2) we set $\tau < 2q - 2p$, and $\tau < q - p$ in the exponents for S_a, and we use (4.3). Consequently, the members in the braces in (4.2) drop out.

Noting this, and setting $\tau - q + p = t$ in the first equality of (4.2), it can be rewritten

$$\begin{aligned} &N_{n0}^{13}(S_a^{(t)}, 0) + X_{na}^{(t)} = 0 \qquad (n = 1, 2, \ldots, 6;\ 0 \leqslant t < q-p) \\ &M_{m0}^{13}(0, T_a^{(\tau)}) + Y_{ma}^{(\tau)} = 0 \qquad (m = 1, 2, 3, 4;\ 0 \leqslant \tau < 2q-2p) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} X_{na}^{(t)} &= N_{n0}^{2}(0, T_a^{(t)}) + RN_{n0}^{*}(0, T_a^{(t-p)}) \\ Y_{ma}^{(\tau)} &= M_{m1}^{13}(0, T_a^{(\tau-q)}) + RM_{m0}^{*}(0, T_a^{(\tau-q)}) \end{aligned} \tag{4.6}$$

The equalities (4.5) are a chain of systems of equations, from which the unknowns

$$T_a^{(\tau)} = (T_{12a}^{(\tau)}, T_{21a}^{(\tau)}, T_{23a}^{(\tau)}, V_{2a}^{(\tau)})$$

$$S_a^{(t)} = (S_{11a}^{(t)}, S_{22a}^{(t)}, S_{33a}^{(t)}, S_{13a}^{(t)}, U_{1a}^{(t)}, U_{3a}^{(t)})$$

can be found successively, in order of growth of τ and t.

This process is an iteration, in which the system

$$M_{m0}^{13}(0, T_a^{(\tau)}) = 0 \qquad (m = 1, 2, 3, 4;\ 0 \leqslant \tau < q) \tag{4.7}$$

consisting of four equations with four unknowns, leads off.

Indeed, (4.7) is obtained from the second equality of (4.5) since $Y_{ma}^{(\tau)} \equiv 0$ for $\tau < q$ because of (4.6) and (4.3). For $\tau \geqslant q$, an inhomogeneous equation expressed by the second equality in (4.5) is obtained for $T_a^{(\tau)}$, but according to (4.6), the $Y_{ma}^{(\tau)}$ therein is expressed in terms of $T_a^{(\tau-q)}$, and upon executing the iteration process $Y_{ma}^{(\tau)}$ must be considered as a known quantity. There is a system of six equations formed by the first group of equalities (4.5) for the six unknowns $S_a^{(t)}$; the quantity $X_{na}^{(t)}$ must hence be considered known since, according to (4.6), it is expressed in terms of T_a.

Besides solutions of the form (4.1), solutions of the form

$$S_b = \sum_{t=0}^{\infty} \varkappa^{-t} S_b^{(t)}, \qquad T_b = \varkappa^{-q+p} \sum_{t=0}^{\infty} \varkappa^{-t} T_b^{(t)} \tag{4.8}$$

can also be constructed.

Proceeding analogously in this case, we obtain the following sequence of systems of equations for the construction of $S_b^{(t)}, T_b^{(t)}$:

$$N_{n0}^{(13)}(S_b^{(\tau)}, 0) + X_{nb}^{(\tau)} = 0 \qquad (n = 1, 2, \ldots, 6;\ 0 \leqslant \tau < 2q - 2p)$$
$$M_{m0}^{13}(0, T_b^{(t)}) + Y_{mb}^{(t)} = 0 \quad (m = 1, 2, 3, 4;\ 0 \leqslant t < q - p) \tag{4.9}$$

where

$$X_{nb}^{(\tau)} = N_{n1}^{13}(S_b^{(\tau-q)}, 0) + RN_{n0}^{*}(S_b^{(\tau-q)}, 0)$$
$$Y_{mb}^{(t)} = M_{m0}^{2}(S_b^{(t)}, 0) + RM_{m0}^{*}(S_b^{(t-p)}, 0) \tag{4.10}$$

A solution of the form (4. 8) can be determined from (4. 9), (4. 10) by using an itera-tion process in which the system

$$N_{n0}^{13}(S_b^{(\tau)}, 0) = 0 \qquad (n = 1, 2, \ldots, 6;\ 0 \leqslant \tau < q) \tag{4.11}$$

consisting of six equations with six unknowns, leads off.

5. Solutions of the form (4. 1) and (4. 8) have a simple physical meaning.

Taking account of (3. 3), (3. 6) and (3. 7), Eqs. (4. 5) and (4. 9) can be written in expanded form as follows:

$$\frac{1}{A_{10}}\frac{\partial S_{11}^{(r)}}{\partial \xi_1} + \frac{\partial S_{13}^{(r)}}{\partial \zeta} + X_1^{(r)} = 0, \quad \frac{1}{A_{10}}\frac{\partial S_{13}^{(r)}}{\partial \xi_1} + \frac{\partial S_{33}^{(r)}}{\partial \zeta} + X_2^{(r)} = 0 \tag{5.1}$$

$$\frac{E}{A_{10}}\frac{\partial U_1^{(r)}}{\partial \xi_1} = S_{11}^{(r)} - \nu(S_{22}^{(r)} + S_{33}^{(r)}) - X_3^{(r)}, \quad 0 = S_{22}^{(r)} - \nu(S_{11}^{(r)} + S_{33}^{(r)}) - X_4^{(r)}$$

$$E\frac{\partial U_3^{(r)}}{\partial \zeta} = S_{33}^{(r)} - \nu(S_{11}^{(r)} + S_{22}^{(r)}) - X_5^{(r)}$$

$$E\left(\frac{1}{A_{10}}\frac{\partial U_3^{(r)}}{\partial \xi_1} + \frac{\partial U_1^{(r)}}{\partial \zeta}\right) = 2(1+\nu)S_{13}^{(r)} - X_6^{(r)}$$

$$\frac{1}{A_{10}}\frac{\partial T_{21}^{(r)}}{\partial \xi_1} + \frac{\partial T_{23}^{(r)}}{\partial \zeta} + Y_1^{(r)} = 0, \qquad \frac{E}{A_{10}}\frac{\partial V_2^{(r)}}{\partial \xi_1} = 2(1+\nu)T_{21}^{(r)} - Y_2^{(r)} \tag{5.2}$$

$$E\frac{\partial V_2^{(r)}}{\partial \zeta} = 2(1+\nu)T_{23}^{(r)} - Y_4^{(r)}, \qquad T_{21}^{(r)} = T_{12}^{(r)} - \frac{Y_3^{(r)} - Y_2^{(r)}}{2(1+\nu)}$$

These systems hold for both quantities with subscript a and with subscript b. In the first case it is necessary to put $r = t$ in (5. 1) and $r = \tau$ in (5. 2), and to consider that $X_{na}^{(t)}, Y_{ma}^{(\tau)}$ are determined by (4. 6). In the second case it is necessary to put $r = \tau$ in (5. 1) and $r = t$ in (5. 2) and to consider that $X_{nb}^{(\tau)}, Y_{mb}^{(t)}$ are determined by (4. 10). The superscripts τ and t should satisfy the inequalities $0 \leqslant \tau < 2q - 2p$ and $0 \leqslant t < q - p$ in both cases.

According to (4. 6) and (4. 10), for $r = \tau < q$ the quantities $X_n^{(\tau)}$ and $Y_m^{(\tau)}$ vanish identically, and the systems (5. 1), (5. 2) become homogeneous, and ξ_2 enters therein only as a parameter. Hence, the following substitution can be made

$$A_{10}\xi_1 = \xi_1' \tag{5.3}$$

Consequently, the systems (5. 1) and (5. 2) pass, respectively, into the equations of the plane and anti-plane problems of elasticity theory in the homogeneous case. In both cases ξ_1', ζ must be considered as Cartesian coordinates, where $S_{ij}^{(\tau)}$, $U_k^{(\tau)}$ play the part of displacements and stresses in the plane problem, and $T_{ij}^{(\tau)}$, $V_k^{(\tau)}$ in the anti-plane problem (ξ_1' corresponds to the subscript 1, and ζ to the subscript 3).

Let us call the solution of the form (4. 1), marked with the additional subscript a, the anti-plane boundary layer, and the solution of the form (4. 8) with the additional subscript

b, the plane boundary layer. The anti-plane boundary layer is understood to be the stress-strain state $(S_a, \quad T_a)$ in which S_a plays a secondary part

$$S_a = O(\varkappa^{-q+p}) T_a$$

and T_a can be determined from the homogeneous (for $Y_m^{(r)} \equiv 0$) equations of the anti-plane problem (5.2), expressed briefly by (4.7), with a formal asymptotic error of the order of $O(\varkappa^{-q})$. Analogously, T_b plays a secondary role in the plane boundary layer $(S_b, \ T_b)$

$$T_b = O(\varkappa^{-q+p}) S_b$$

and the homogeneous (for $X_n^{(r)} \equiv 0$) system of equations of the plane problem (5.1) expressed briefly by (4.11) and defining S_b with a formal asymptotic error of the order of $O \ (\varkappa^{-q})$, leads off.

The connection of the boundary layer to the plane and anti-plane problems of elasticity theory was indicated in a number of papers [3–9]. Let us examine the physical meaning of this result.

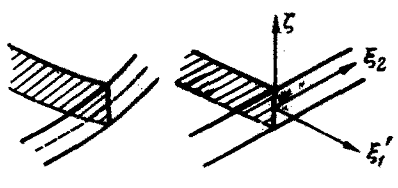

Fig. 1

Let the side surface of the shell pass through the line $\alpha_1 = 0$. Let us draw a shell cross section in the planes of the normal to the surface and of the tangential normal to the line $\alpha_1 = 0$. through some point of this line. The results being discussed mean that within the scope of the formal asymptotic error $\varkappa^{-q} = h_*$ the boundary layer in this section is independent of what occurs in other normal sections, and in constructing a boundary layer within the same accuracy the shell can be replaced by an elastic layer (Fig. 1) $0 \leqslant \xi_1' < -\infty, \quad -\infty < \xi_2 < +\infty, \quad -1 \leqslant \zeta \leqslant +1$

each of whose cross sections $\xi_2 = \text{const}$ behaves the same as the shell cross section under consideration, which is shown shaded in Fig. 1. It is seen from (5.3) that the length scale will be distorted along the cross sections and also differently in different sections, in general.

6. Of interest later in boundary layer theory is the boundary value problem consisting of integrating (5.1), (5.2) in the infinite half-strip

$$-1 \leqslant \zeta = \alpha_3 / h \leqslant +1, \quad \xi_1' = A_{10}\alpha_1 / h \leqslant 0 \tag{6.1}$$

while taking into account:

a) boundary conditions on the endface $\xi_1' = 0$ or equilvalently $\alpha_1 = 0$, about which no assumptions will be made yet;

b) boundary conditions on the face surfaces $\zeta = \pm 1$, consisting of the requirement of no stresses at all (it is assumed that the loading applied to the face surfaces has been taken into account in constructing the interior state of stress);

c) the requirement of damping of all boundary layer stresses and displacements as $\xi_1' \to -\infty$.

Taking account of (1.2), (1.3), we write the boundary condition (b) and the requirement (c) as follows: $S_{13} = T_{23} = S_{33} = 0 \qquad (\zeta = \pm 1) \qquad (6.2)$

$$S_{11} = T_{12} = S_{13} = 0, \quad U_1 = V_2 = U_3 = 0 \quad (\xi_1' \to -\infty) \tag{6.3}$$

This means that the inhomogeneous equations (5.1), (5.2) of the plane and anti-plane problems of elasticity theory must be solved in the half-strip (6.1) by considering the face sides of the half-strip and its infinitely remote endface to be unloaded. Meanwhile, the quantities $X_1^{(r)}$, $X_2^{(r)}$, $Y_1^{(r)}$ are mass force components, and therefore they should be in equilibrium with the active and reactive forces applied to the endface $\xi_1' = 0$, i.e. should satisfy the following four conditions for equilibration of the half-strip as a whole:

$$\begin{gathered}
\int_{-1}^{+1} S_{11}^{(r)}|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} X_1^{(r)} A_{10}\, d\xi_1 = 0 \\
\int_{-1}^{+1} T_{21}^{(r)}|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} Y_1^{(r)} A_{10}\, d\xi_1 = 0 \\
\int_{-1}^{+1} S_{13}^{(r)}|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} X_2^{(r)} A_{10}\, d\xi_1 = 0 \\
\int_{-1}^{+1} S_{11}^{(r)}|_{\alpha_1=0} \zeta\, d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} [\zeta X_1^{(r)} - A_{10}\xi_1 X_2^{(r)}]\, A_{10}\, d\xi_1 = 0
\end{gathered} \tag{6.4}$$

The equalities (6.4) hold for both the plane and anti-plane boundary layers. They will be called the static damping conditions. It is completely evident that (6.4) are necessary for the existence of a damping boundary layer, i.e. satisfying condition (6.3). It will henceforth be assumed that (6.4) are also sufficient for this.

The assumption of sufficiency of the static damping conditions results from the Saint-Venant principle. The requirements (6.2), (6.3) and three conditions on the endface $\xi_1' = 0$ must be taken into account in constructing the boundary layer. An over-determined problem is obtained in which the first three equalities in (6.3) can be considered redundant since it is evident that a solution always exists in which the stresses at infinity tend to a finite nonzero limit. Let such a solution satisfy the static damping conditions for any values of r. But as has been shown above, for $r < q$ Eqs. (5.1), (5.2) become the homogeneous equations of the plane and anti-plane problems, and then Eqs. (6.4) pass into the equilibration conditions for all the forces applied to the half-strip, except possibly the reactions at infinity. In conformity with the Saint-Venant principle, the stress-strain state $(S^{(r)}, T^{(r)})$ will be a damping state for such r. For $r = \rho \geqslant q$ the quantities $X_1^{(\rho)}$, $X_2^{(\rho)}$, $Y_1^{(\rho)}$ are nonzero. However, by using (4.6), (4.10) they are expressed in terms of $(S^{(r)}, T^{(r)})$, where $r < \rho$, and therefore, damp out rapidly themselves. Again applying the Saint-Venant principle to a strip loaded by forces on the endface $\xi_1' = 0$ and the mass forces $X_1^{(\rho)}$, $X_2^{(\rho)}$, $Y_1^{(\rho)}$ localized near $\xi_1' = 0$, and making use of the method of induction, it can be asserted that $(S^{(\rho)}, T^{(\rho)})$ will even damp for any ρ upon compliance with the conditions (6.4).

7. The static damping conditions (6.4) can also be written in expanded form. If we speak of the anti-plane boundary layer, then subscripts a must be appended to the quantities $T^{(r)}$, $S^{(r)}$, $X_n^{(r)}, Y_m^{(r)}$, and we must set $r = \tau$ in the formulas for $Y_{ma}^{(r)}$ and must disclose the meaning of $X_{na}^{(t)}$, $Y_{ma}^{(\tau)}$ by using (4.6), (3.6)–(3.10), analogously for the plane boundary layer. Two groups of equalities are hence obtained, which can be called the static damping conditions for the anti-plane and plane sublayers, respectively. They are valid for any values of the superscripts t and τ, but are awkward, and will not henceforth be needed in such general form. Meanwhile, if the maximum admissible values of

t and τ are suitably constrained, then the equalities under discussion are simplified considerably and become the following:

static damping conditions for the anti-plane boundary layer

$$\int_{-1}^{+1} S_{11a}^{(t)}\,|_{\alpha_1=0} d\zeta = 0 \qquad (\tau < q)$$

$$\int_{-1}^{+1} T_{12a}^{(\tau)}\,|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \frac{R}{R_{20}} T_{23a}^{(\tau-q)} A_{10} d\xi_1 = 0 \qquad (\tau < 2q)$$

$$\int_{-1}^{+1} S_{13a}^{(t)}\,|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} [d_{20} T_{23a}^{(t)} + R k_{10} T_{23a}^{(t-p)}]\, A_{10} d\xi_1 = 0 \qquad (\tau < q) \qquad (7.1)$$

$$\int_{-1}^{+1} S_{11a}^{(t)}\,|_{\alpha_1=0}\, \zeta\, d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \{\zeta\, [d_{20} T_{12a}^{(t)} + 2 R k_{10} T_{12a}^{(t-p)}] -$$

$$- A_{10}\xi_1 [d_{20} T_{23a}^{(t)} + R k_{10} T_{23a}^{(t-p)}]\}\, A_{10} d\xi_1 = 0 \qquad (\tau < q)$$

static damping conditions for the plane boundary layer

$$\int_{-1}^{+1} S_{11b}^{(\tau)}|_{\alpha_1=0}\, d\zeta - \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} R k_{20} S_{22b}^{(\tau-q)} A_{10} d\xi_1 = 0 \qquad (\tau < 2q)$$

$$\int_{-1}^{+1} T_{12b}^{(t)}\,|_{\alpha_1=0} d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} [d_{20} S_{22b}^{(t)} + k_{10} S_{22b}^{(t-p)}]\, A_{10} d\xi_1 = 0 \qquad (t < q) \qquad (7.2)$$

$$\int_{-1}^{+1} S_{13b}^{(\tau)}\,|_{\alpha_1=0} d\zeta - \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \frac{R}{R_{20}} S_{22b}^{(\tau-q)} A_{10} d\xi_1 = 0 \qquad (\tau < 2q)$$

$$\int_{-1}^{+1} S_{11b}^{(\tau)}\,|_{\alpha_1=0}\, \zeta\, d\zeta + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} R \left(\frac{A_{10}\xi_1}{R_{20}} - k_{20}\zeta\right) S_{22b}^{(\tau-q)} A_{10} d\xi_1 = 0 \qquad (\tau < 2q)$$

These simplifications are based on the following considerations:

1) if $r < q$, then the quantities $S_b^{(r)}$ satisfy the homogeneous equations (5.1), and the quantities $T_a^{(r)}$ are homogeneous equations (5.2). In particular, this latter means that

$$T_{12a}^{(r)} = T_{21a}^{(r)} \qquad (r < q) \qquad (7.3)$$

2) in evaluating the integrals in the static damping conditions, it can be considered that

$$A_{10},\ k_{10},\ k_{20},\ R_{10},\ R_{20},\ d_{20} = \text{const} \qquad (7.4)$$

since all these quantities introduced in Sect. 3 are independent of ξ_1 and ζ

3) the formula

$$\zeta \frac{R}{R_{20}} d_3 T_{23a}^{(r)} + \frac{2R}{R_{20}} T_{23a}^{(r)} = \frac{R}{R_{20}} \frac{1}{\zeta} \frac{\partial}{\partial \zeta} (\zeta^2 T_{23a}^{(r)}) \qquad (7.5)$$

is valid, as is easily verified by taking into account that d_3 is expanded by using (3.4), and since $T_{23a}^{(r)}$ satisfies the second condition (6.2), then from (7.5) there follow

$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \left[\zeta \frac{R}{R_{20}} d_3 T_{23a}^{(r)} + \frac{2R}{R_{20}} T_{23a}^{(r)}\right] A_{10}\, d\xi_1 = \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \frac{R}{R_{20}} T_{23a}^{(r)} A_{10}\, d\xi_1 \qquad (7.6)$$

$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \left[\zeta \frac{R}{R_{20}} d_3 T_{23a}^{(r)} + \frac{2R}{R_{20}} T_{23a}^{(r)}\right] \zeta A_{10}\, d\xi_1 = 0$$

Moreover the formula
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n} DT_{12a}^{(r)} A_{10} d\xi_1 = 0 \qquad (r<q) \tag{7.7}$$
holds, in which D is an expression independent of ξ_1, ζ, and n is a nonnegative number. In fact, integrating by parts with respect to ξ_1, and considering that $\xi_1^{n+1} T_{12a}^{(r)}$ vanishes at infinity, we obtain
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n} DT_{12a}^{(r)} A_{10}\, d\xi_1 = -\frac{1}{n+1} \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n+1} D \frac{\partial T_{12a}^{(r)}}{\partial \xi_1} A_{10}\, d\xi_1 \tag{7.8}$$
But the first equation of (5.2) is homogeneous for the assumed values of r; hence, expressing $T_{12a}^{(r)}$ in terms of $T_{23a}^{(r)}$ with its aid, we have
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n} DT_{12a}^{(r)} A_{10}\, d\xi_1 = \frac{A_{10}}{n+1} \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n+1} D \frac{\partial T_{23a}^{(r)}}{\partial \zeta} A_{10}\, d\xi_1$$
The integral on the right side of this equality vanishes because of the second condition of (6.2).

It can be considered in (7.7) that
$$D = Ed_{11} = ER\left(\frac{1}{A_1}\right)_1 \frac{\partial}{\partial \xi_1} \qquad (E \text{ is independent of } \xi_1, \zeta) \tag{7.9}$$
The proof of (7.7) then simplifies since the integral on the right side of (7.8) will be considered.

The formula
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n} DS_{11b}^{(r)} A_{10}\, d\xi_1 = \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1^{\,n} DS_{13b}^{(r)} A_{10}\, d\xi_1 = 0 \tag{7.10}$$
in which D is independent of ξ_1, ζ or has the form (7.9), can be deduced analogously. The equalities (7.10), (7.9) are proved exactly as are (7.10), (7.9), except the first or second of equations (5.1) must, respectively, be utilized in place of the first equation in (5.2).

Finally, the formulas
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \zeta S_{11b}^{(r)} A_{10}\, d\xi_1 = 0, \qquad \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \xi_1 \zeta d_{11} S_{11b}^{(r)} A_{10}\, d\xi_1 = 0 \tag{7.11}$$
are valid.

They are both proved similarly. If, for definiteness, we speak of the first equality, it results from the computations presented below in which integration by parts with respect to ξ_1 and ζ is performed successively, and the first two equalities in (5.1) and conditions (6.2), (6.3) are also utilized
$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \zeta S_{11b} A_{10}\, d\xi_1 = -\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10} \xi_1 \zeta \frac{\partial S_{11b}}{\partial \xi_1} d\xi_1 =$$
$$= \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10} \xi_1 \zeta \frac{\partial S_{13b}}{\partial \zeta} A_{10} d\xi_1 = -\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10} \xi_1 S_{13b} A_{10} d\xi_1 =$$
$$= \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \frac{A_{10}}{2} \xi_1^{2} \frac{\partial S_{13b}}{\partial \xi_1} A_{10} d\xi_1 = -\frac{1}{2} \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10}^{2} \xi_1^{2} \frac{\partial S_{33b}}{\partial \zeta} A_{10} d\xi_1 = 0$$
The simplifications which are made in the static damping conditions (7.1), (7.2)

also result from (7.3)–(7.11).

8. Now, let us deduce the kinematic damping conditions for certain cases. Let it be required to integrate (5.1) in the rectangle

$$0 \geqslant \xi_1' = A_{10}\alpha_1 / h \geqslant -l, \quad -1 \leqslant \zeta \leqslant +1 \tag{8.1}$$

by satisfying the boundary conditions

$$S_{13} = S_{33} = 0 \quad (\zeta = \pm 1) \qquad EU_1 = EU_3 = 0 \quad (\xi_1' = -l) \tag{8.2}$$

corresponding to the requirements of no exterior forces on the face surfaces, and rigid framing at the endface $\xi_1' = -l$; moreover, let the conditions

$$EU_1 = \sum_{i=0}^{n} a_i\zeta^i, \qquad EU_3 = \sum_{k=0}^{m} b_k\zeta^k \tag{8.3}$$

be posed on the endface $\xi_1' = 0$.

It is assumed that the superscript r in (5.1) can be arbitrary (to simplify the notation it has not been written down).

Making use of the linearity of the problem, we represent its solution as

$$S = S^* + \sum_{i=0}^{n} a_i S^{[ui]} + \sum_{k=0}^{m} b_k S^{[wk]}$$

Here S^*, $S^{[ui]}$, $S^{[wk]}$ are solutions of (5.1) satisfying conditions (8.2). The first of these solutions corresponds to inhomogeneous (for the $X_n^{(r)}$ not vanishing simultaneously) equations (5.1) and homogeneous conditions on the endface $\xi_1' = 0$

$$EU_1^* |_{\xi_1'=0} = 0, \qquad EU_3^* |_{\xi_1'=0} = 0$$

and the second and third solutions correspond to homogeneous equations (5.1) and, correspondingly, the following inhomogeneous conditions on the endface $\xi_1' = 0$:

$$EU_1^{[ui]} = \zeta^i, \qquad EU_3^{[ui]} = 0 \text{ and } EU_1^{[wk]} = 0, \qquad EU_3^{[wk]} = \zeta^k$$

The horizontal and vertical reactive forces P_1, P_2 and the reactive moment P_3 originating at the endface $\xi_1' = -l$ correspond to each of the listed states of stress. Let us mark them with the same symbols as the state of stress generating them. Then by requiring that the desired state of stress be given in the framing $\xi_1' = -l$, we obtain the zero reactions P_1, P_2, P_3

$$P_\rho^* + \sum_{i=0}^{n} a_i P_\rho^{[u,i]} + \sum_{k=0}^{m} b_k P_\rho^{[w,k]} = 0 \qquad (\rho = 1, 2, 3) \tag{8.4}$$

and these equalities are the kinematic damping conditions for the problem under consideration.

Indeed, let the superscript r in (5.1) be so small that these equations are homogeneous. Then self-equilibration of the stresses on the endface $\xi_1' = 0$ will result from the absence of reactions on the endface $\xi_1' = -l$, and as a consequence of the Saint-Venant principle, damping of the solution of the problem under discussion will hold. Reasoning further as in Sect. 6, it is easy to see that compliance with (8.4) will assure damping for any r.

The parameter l, the length of the rectangle (8.1), certainly enters the damping conditions. However, it is clear from physical considerations that for sufficiently large l (as compared to the width of the rectangle, i.e. with two), the equalities (8.4) depend slightly on l, and the rectangle (8.1) can be considered as a half-strip.

9. Now, let (5.1) be integrated in the rectangle (8.1) while again satisfying the condition (8.2), but the following mixed conditions are posed on the endface $\xi_1' = 0$ instead of (8.3)

$$S_{11}|_{\xi_1'=0} = f(\zeta), \qquad EU_3|_{\xi_1'=0} = \sum_{k=0}^{m} c_k \zeta^k \tag{9.1}$$

where $f(\zeta)$ is an arbitrary function.

The first of these equalities gives the boundary values of the stress S_{11}. These values must be subject to the first and fourth equalities in (6.4). Hence, the damping conditions will coincide with the static damping conditions in the mixed problem under consideration.

Let S^*, $S^{[s]}$, $S^{(wk)}$ denote the solutions of (5.1) satisfying the conditions (8.2). The first is obtained as a result of solving the inhomogeneous equations, and the remaining two, as a result of solving the homogeneous equations. They should satisfy the conditions

$$S_{11}|_{\xi_1'=0} = 0,\ f(\zeta),\ 0; \qquad EU_3|_{\xi_1'=0} = 0,\ 0,\ \zeta^k$$

respectively, on the endface $\xi_1' = 0$ (the quantity $S^{(wk)}$, is different in meaning from the quantity $S^{[wk]}$ introduced above).

Let each of these states of stress yield the vertical reaction P_2 with appropriate additional indices at the endface $\xi_1' = -l$. Then requiring that the reaction P_2 not be present in the total solution, we obtain

$$P_2^* + P_2^{(s)} + \sum_{k=0}^{m} c_k P_2^{(wk)} = 0 \tag{9.2}$$

But the longitudinal force and moment due to the edge stresses S_{11} and the damped mass forces are mutually equilibrated near the endface $\xi_1' = 0$ because of the static damping conditions and in order for all the forces to be self-equilibrated near $\xi_1' = 0$ it is sufficient to require compliance with the equality (9.2), which is the single kinematic damping condition for the mixed problem (9.1). Exactly as has been done in Sects. 6 and 8, it is easy to show that the first and fourth static damping condition (6.4), together with the kinematic damping condition (9.2), assure a decrease in the solution of (5.1) in the mixed problem under consideration.

10. A theory of the interior state of stress, the crux of which is the following, has been constructed in [1].

Let us assume that an elastic medium is referred to the tri-orthogonal coordinate system (1.1), and let σ_{st} and v_s denote the stresses and displacements of the interior state of stress.

It is assumed in [1] that the nonsymmetric stresses introducible by using the formulas

$$\begin{aligned} s_{ij} &= (1 + \alpha_3 / R_k)\sigma_{ij} \qquad (i = 1,2;\ j \neq k = 1,2) \\ s_{3i} &= s_{i3} = (1 + \alpha_3 / R_j)\sigma_{i3} \qquad (i \neq j = 1,2) \\ s_{33} &= (1 + \alpha_3 / R_1)\ (1 + \alpha_3 / R_2)\ \sigma_{33} \end{aligned} \tag{10.1}$$

and the displacements v_s possess asymptopic properties expressed by the equalities

$$s_{ij} = \varkappa^{q+d} s_{ij}^{\cdot},\ s_{i3} = \varkappa^{p+d} s_{i3}^{\cdot},\ s_{33} = \varkappa^{c+d} s_{33}^{\cdot},\ v_i = \varkappa^{q-p+d} v_i^{\cdot},\ v_3 = \varkappa^{q-c+d} v_3^{\cdot} \tag{10.2}$$

Here the number d characterizes the intensity of the exterior effects, and is selected as a function of the conditions of the problem; the quantities with the dots are series in descending powers of $\varkappa$ starting with $\varkappa^\circ$, for example

$$s_{ij}^{\cdot} = \sum_{l=0}^{L} \varkappa^{-l} s_{ij}^{(l)} \tag{10.3}$$

The numbers p, q have the same meaning as in Sect. 3, i.e. p/q equals the index of variability of t, and the number c is defined by

$$c = 0 \quad \text{for } 2p \leqslant q, \qquad c = 2p - q \quad \text{for } 2p \geqslant q \tag{10.4}$$

in which the well-known fact, that the properties of a state of stress change substantially when t passes through the value $t = {}^1/_2$, is expressed.

A theory of the interior state of stress has been constructed with a formal asymptotic error of the order of (4.4), i.e. the upper limit of the summation L was taken to equal $2q - 2p - 1$.

Within the span of such accuracy the theory of the interior state of stress is equivalent to some modification of classical shell theory. Here, unless otherwise specified, the interior state of stress is understood to be the state of stress possessing the property (10.2), and constructed with a formal asymptotic error (4.4). Moreover, it is always assumed that

$$t < 1, \quad \text{i.e.} \quad p < q \tag{10.5}$$

The law of variation of stresses and displacements over the shell thickness is defined for the interior state of stress by the formulas

$$\begin{gathered} s_{ij} = s_{ij0} + \zeta s_{ij1}, \qquad s_{i3} = s_{i30} + \zeta s_{i31} + \zeta^2 s_{i32} \quad (i, j = 1,2) \\ s_{33} = s_{330} + \zeta s_{331} + \zeta^2 s_{332} + \zeta^3 s_{333} \qquad (i, j = 1, 2, 3) \\ v_k = v_{k0} + \zeta v_{k1} \quad (\zeta = \alpha_3/h) \qquad (k = 1, 2) \end{gathered} \tag{10.6}$$

in which the quantities with additional subscripts are functions of the two variables α_1, α_2. Formulas (10.2) and (10.3) generally remain valid for them; for example

$$s_{ij0} = \varkappa^{q+d} s_{ij0}^{\cdot}, \qquad s_{ij0}^{\cdot} = \sum_{l=0}^{L} \varkappa^{-l} S_{ij0}^{(l)}$$

However, some of the quantities in (10.6) vanish for not too large l. Namely

$$v_{i0}^{(l)} = s_{ij1}^{(l)} = s_{i32}^{(l)} \equiv 0 \qquad (i, j = 1, 2; \quad 0 \leqslant l < q - 2p + c) \tag{10.7}$$

Moreover, the following formulas are valid

$$\begin{gathered} v_{i1}^{(l)} = -\varphi_{i0}^{(l-q+2p-c)} \quad (i = 1, 2; \quad q - 2p + c \leqslant l < 2q - 2p) \\ \varphi_{i0}^{(l)} = -R\left(\frac{\varkappa^{-p}}{A_i}\frac{\partial v_{30}^{(l)}}{\partial \alpha_1} + \frac{v_{i0}^{(l-2p+c)}}{R_i}\right) \quad (i = 1, 2; \quad 0 \leqslant l < 2q - 2p) \end{gathered} \tag{10.8}$$

In concluding the section, let us refine some of the results of [1] elucidated here.

In the notation accepted, Hooke's law for the deformation of a transverse elongation is expressed by the equality

$$E\left(1 + \zeta\frac{h}{R_1}\right)\left(1 + \zeta\frac{h}{R_2}\right)h^{-1}\frac{\partial v_3}{\partial \zeta} = s_{33} - \nu\left(1 + \zeta\frac{h}{R_1}\right)s_{11} - \nu\left(1 + \zeta\frac{h}{R_2}\right)s_{22} \tag{10.9}$$

Let us consider formulas (10.2)–(10.4) and (10.6) to remain valid for nonsymmetric stress components and for the displacements v_1, v_2, where $L = 2q - 2p - 1$ in

(10.3), but we take a more exact expression for the displacement v_3

$$v_3 = \varkappa^{q-c+d} \sum_{l=0}^{3q-2p-1} \varkappa^{-l} (v_{30}^{(l)} + \zeta v_{31}^{(l)} + \zeta^2 v_{32}^{(l)}) \tag{10.10}$$

Inserting the expansions (10.2), (10.3), (10.6) and (10.10) into (10.9), taking (3.2) into account, and equating coefficients of identical powers of $\varkappa$, we obtain the equation

$$\left[\frac{E}{R} v_{31}^{(l)} - s_{330}^{(l-2q+2c)} + \nu (s_{110}^{(l-q+c)} + s_{220}^{(l-q+c)})\right] + \tag{10.11}$$

$$+ \left\{\frac{2E}{R}\left[v_{32}^{(l)} + \frac{R}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_{31}^{(l-q)}\right] - s_{331}^{(l-2q+2c)} + \nu (s_{111}^{(l-q+c)} + s_{221}^{(l-q+c)}) + \right.$$

$$\left. + \nu R\left(\frac{s_{110}^{(l-2q+c)}}{R_1} + \frac{s_{220}^{(l-2q+c)}}{R_2}\right)\right\}\zeta + \cdots = 0 \qquad (0 \leqslant l < 3q - 2p)$$

in which terms containing ζ in powers above the first are denoted by dots.

Here and henceforth, it is assumed throughout that (4.3) remains valid for quantities referring to the interior state of stress, i.e. they are also nonzero only for nonnegative values of the superscript.

Requiring that the coefficient of ζ^0 vanish in (10.11), we obtain

$$v_{31}^{(l)} = -\nu \frac{R}{E} (s_{110}^{(l-q+c)} + s_{220}^{(l-q+c)}) \qquad (0 \leqslant l < 2q - 2c) \tag{10.12}$$

The upper limit of admissible values l changes in this equality, which is legitimate since $2q - 2c \leqslant 3q - 2p$.

Analogously, equating coefficients of ζ^1 in (10.11) and utilizing (10.12), we obtain

$$v_{32}^{(l)} = \frac{R}{2E}\left[-\nu (s_{111}^{(l-q+c)} + s_{221}^{(l-q+c)}) + \right. \tag{10.13}$$

$$\left. + \nu R\left(\frac{s_{110}^{(l-2q+c)}}{R_2} + \frac{s_{220}^{(l-2q+c)}}{R_1}\right) + s_{331}^{(l-2q+2c)}\right] \qquad (0 \leqslant l < 3q - 2p)$$

11. The question of interaction between the interior state of stress and the boundary layer will be considered in subsequent sections. Let us elucidate the method of this investigation.

Let us assume that the total state of stress of a shell is a linear combination of the interior state of stress and the anti-plane and plane boundary layers, namely, the following formulas hold:

$$s_{ij}^* = s_{ij} + S_{ija} + S_{ijb}, \qquad v_k^* = v_k + hU_{ka} + hU_{kb} \tag{11.1}$$

or

$$s_{ij}^* = s_{ij} + T_{ija} + T_{ijb}, \qquad v_k^* = v_k + hV_{ka} + hV_{kb} \tag{11.2}$$

where v_k^*, s_{ij}^* are the total displacements and total nonsymmetric stresses (Sect. 10).

Let us take these expansions

$$S_a = \varkappa^{\lambda-q+p} \sum_{l=0}^{\infty} \varkappa^{-l} S_a^{(l)}, \qquad T_a = \varkappa^{\lambda} \sum_{l=0}^{\infty} \varkappa^{-l} T_a^{(l)}$$

$$S_b = \varkappa^{\mu} \sum_{l=0}^{\infty} \varkappa^{-l} S_b^{(l)}, \qquad T_b = \varkappa^{\mu-q+p} \sum_{l=0}^{\infty} \varkappa^{-l} T_b^{(l)} \tag{11.3}$$

for the quantities comprising the plane and anti-plane boundary layers.

They differ from the expansions (4.1) and (4.8) in that the quantities marked with the subscripts a and b are increased, respectively, $\varkappa^{\lambda}$ and $\varkappa^{\mu}$ times. This is legitimate since $(S_a^{(t)}, T_a^{(t)})$ and $(S_b^{(t)}, T_b^{(t)})$ are defined separately as solutions of some homogeneous equations on which only the homogeneous conditions (6.2), (6.3) have as yet been imposed.

The numbers λ, μ characterize the intensities of the anti-plane and plane boundary layers. They should be selected as a function of the kind of boundary conditions of elasticity theory which must be satisfied on the side surfaces of the shell. The following is what is meant.

Let λ, μ be fixed integers. Then the stresses and displacements defined by (11.1) and (11.2) can be represented as expansions in descending powers of $\varkappa$. To do this it is necessary to express h in terms of $\varkappa$ therein by using (3.2), and to use the expansions (10.2), (10.3) for quantities connected with the interior state of stress, and the expansions (11.3) for quantities connected with the boundary layers.

Substituting these expansions in the triple of three-dimensional boundary conditions $\Gamma = 0$ on the side surfaces, and equating coefficients of identical powers of $\varkappa$ in each condition, we obtain a sequence of boundary conditions $\Gamma_s = 0$. A certain sequence of such static or kinematic damping conditions $\Omega_s = 0$ must be appended to these relationships, whereby only the damped solution of the boundary layer equations will drop out in the total state of stress. The set of boundary conditions $\Gamma_s = 0$ and $\Omega_s = 0$ defines some iteration process for satisfying the boundary conditions, in which the arbitrary values contained in the interior state of stress equations and the boundary layer equations should be used successively. In general, i.e. for arbitrarily chosen λ, μ, the order and type of these equations will contradict the structure $\Gamma_s = 0$ and $\Omega_s = 0$, and the problem is to select values of λ, μ which do not result in such an inconsistency.

12. Let the shell have a free edge, which coincides with some side surface passing through the line $\alpha_1 = 0$. Then by virtue of (11.1) and (11.2) it must be required that the following boundary conditions be satisfied

$$s_{11}^* \equiv s_{11} + S_{11a} + S_{11b} = 0, \quad s_{12}^* = s_{12} + T_{12a} + T_{12b} = 0$$
$$s_{13}^* = s_{13} + S_{13a} + S_{13b} = 0 \qquad (\alpha_1 = 0)$$

(homogeneity of the boundary conditions is utilized, and symmetric stresses are replaced by nonsymmetric ones).

Let us assume that consistent values of λ, μ are written thus in this case (*)

$$\lambda = 2p - c + d, \qquad \mu = p + d \tag{12.1}$$

Let us replace the s_{ij} in the boundary conditions by the expansions (10.2), (10.3), (10.6), and the S_{ija}, S_{ijb}, T_{ija}, T_{ijb} by the expansions (11.3), and let us equate coefficients of identical powers of $\varkappa$ to zero. We obtain the following sequence of boundary conditions

$$\begin{gathered} s_{110}^{(l)} + \zeta s_{111}^{(l)} + S_{11a}^{(l-2q+3p-c)} + S_{11b}^{(l-q+p)} = 0 \\ s_{120}^{(l)} + \zeta s_{121}^{(l)} + T_{12a}^{(l-q+2p-c)} + T_{12b}^{(l-2q+2p)} = 0 \\ s_{130}^{(l)} + \zeta s_{131}^{(l)} + \zeta^2 s_{132}^{(l)} + S_{13a}^{(l-q+2p-c)} + S_{13b}^{(l)} = 0 \\ (\alpha_1 = 0, \quad 0 \leqslant l < 2q - 2p) \end{gathered} \tag{12.2}$$

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*) The validity of the assumptions (12.1), (13.1), (14.2) is discussed in Sect. 16.

Values of the superscript l therein are constrained by the inequalities in the parentheses since the formulas (10.6) used here are valid just for such l.

Besides (12.2), compliance with the four static damping conditions must be required in the case under consideration. It is convenient to write them as follows for future reasoning:

$$2s_{110}^{(l)}|_{\alpha_1=0}=0, \qquad 2s_{120}^{(l)}|_{\alpha_1=0}=0$$

$$2\left(s_{130}^{(l)}+\frac{1}{3}s_{132}^{(l)}\right)_{\alpha_1=0}-\int_{-1}^{+1}d\zeta\int_{-\infty}^{0}[d_{20}T_{23a}^{(l-q+2p-c)}+Rk_{10}T_{23a}^{(l-q+p-c)}]A_{10}d\xi_1+$$
$$+\int_{-1}^{+1}d\zeta\int_{-\infty}^{0}\frac{R}{R_{20}}S_{22b}^{(l-q)}A_{10}d\xi_1=0 \qquad (12.3)$$

$$\frac{2}{3}s_{111}^{(l)}|_{\alpha_1=0}-\int_{-1}^{+1}d\zeta\int_{-\infty}^{0}[\zeta d_{20}T_{21a}^{(l-2q+3p-c)}-A_{10}\xi_1 d_{20}T_{23a}^{(l-2q+3p-c)}]A_{10}d\xi_1=0$$

$$(0\leqslant l<2q-2p)$$

In order to obtain this result, let us integrate each of the boundary conditions (12.2) with respect to ζ in the band $(-1,+1)$, let us multiply the first equality (12.2) by ζ and let us again integrate with respect to ζ in the range $(-1,+1)$. Then recalling that $s_{ijk}^{(l)}$ are independent of ζ, we obtain the following four equalities:

$$2s_{110}^{(l)}|_{\alpha_1=0}+\int_{-1}^{+1}[S_{11a}^{(l-2q+3p-c)}+S_{11b}^{(l-q+p)}]_{\alpha_1=0}\,d\zeta=0$$

$$2S_{120}^{(l)}|_{\alpha_1=0}+\int_{-1}^{+1}[T_{12a}^{(l-q+2p-c)}+T_{12b}^{(l-2q+2p)}]_{\alpha_1=0}\,d\zeta=0 \qquad (12.4)$$

$$2\left[s_{130}^{(l)}+\frac{1}{3}s_{132}^{(l)}\right]_{\alpha_1=0}+\int_{-1}^{+1}\left[S_{13a}^{(l-q+2p-c)}+S_{13b}^{(l)}\right]_{\alpha_1=0}d\zeta=0$$

$$\frac{2}{3}s_{111}^{(l)}|_{\alpha_1=0}+\int_{-1}^{+1}[S_{11a}^{(l-2q+3p-c)}+S_{11b}^{(l-q+p)}]_{\alpha_1=0}\,\zeta d\zeta=0$$

$$(0\leqslant l<2q-2p)$$

Let us eliminate the integrals in the left side of (12.4) by using (7.1) and (7.2) and let us discard quantities whose superscripts are known to be negative because of the inequalities constraining l in (12.2). This will indeed lead to the required result.

Thus, we have the three equalities (12.2) and the four equalities (12.3). They will all be satisfied if the first and third equalities of (12.2) are considered as endface conditions for the plane boundary layer, the second equality of (12.2) as endface conditions for the anti-plane boundary layer, and the four equalities of (12.3) as boundary conditions for the interior state of stress.

Hence, admitting a formal asymptotic error of the order of

$$\varepsilon_b=O(\varkappa^{-q+2p-c}) \qquad (12.5)$$

in the construction of the plane boundary layer, the endface conditions for the plane boundary layer can be reduced to the form (12.6)

$$S_{11b}^{(r)}=0, \qquad S_{13b}^{(r)}=(3\zeta^2-1)s_{130}^{(r)}-\zeta s_{131}^{(r)} \qquad (\alpha_1=0,\ 0\leqslant r<q-2p+c)$$

and admitting a formal asymptotic error of the order of

$$\varepsilon_a = O(\varkappa^{-q+p}) = O(h_*^{1-t}) \qquad (12.7)$$

for the anti-plane boundary layer, the endface condition can be written as:

$$T_{12a}^{(r)} = -\zeta s_{121}^{(r+q-2p+c)} \qquad (0 \leqslant r \leqslant q-p) \qquad (12.8)$$

Proceeding to the proof of the validity of (12.6) and (12.8), we assume the superscript l in the first two equalities of (12.2) to be constrained, respectively, by the inequalities

$$0 \leqslant l < q-p, \qquad 0 \leqslant l < q-2p+c$$

Then by virtue of (4.3), all the boundary-layer quantities vanish and the equalities

$$s_{110}^{(l)} + \zeta s_{111}^{(l)} = 0 \quad (0 \leqslant l < q-p), \qquad s_{120}^{(l)} + \zeta s_{121}^{(l)} = 0 \quad (0 \leqslant l < q-2p+c)$$
$$(\alpha_1 = 0) \qquad (12.9)$$

are obtained which will be a result of the static damping conditions (12.3), as can easily be seen by taking account of (4.3), (10.4) and (10.7). Hence, it can be considered that the superscript l in the first two equalities of (12.2) is constrained, respectively, by the inequalities $\quad q-p \leqslant l < 2q-2p, \quad q-2p+c \leqslant l < 2q-2p$

Taking this into account, we make the corresponding substitutions for the superscript l in the inequalities in (12.2)

$$l-q+p = r, \qquad l-q+2p-c = r, \qquad l = r$$

We obtain

$$s_{110}^{(r+q-p)} + \zeta s_{111}^{(r+q-p)} + S_{11a}^{(r-q+2p-c)} + S_{11b}^{(r)} = 0 \quad (0 \leqslant r < q-p)$$
$$s_{120}^{(r+q-2p+c)} + \zeta s_{121}^{(r+q-2p+c)} + T_{12a}^{(r)} + T_{12b}^{(r-q+c)} = 0 \quad (0 \leqslant r < q-c) \qquad (12.10)$$
$$s_{130}^{(r)} + \zeta s_{131}^{(r)} + \zeta^2 s_{132}^{(r)} + S_{13a}^{(r-q+2p-c)} + S_{13b}^{(r)} = 0 \quad (0 \leqslant r < 2q-2p)$$
$$(\alpha_1 = 0)$$

It is easy to verify (*) that $q-c \geqslant q-p$. Hence, considering it sufficient to construct the anti-plane boundary layer with the formal asymptotic error (12.7), the term associated with the plane boundary layer can be discarded in the second equality in (12.10). Equivalently, terms associated with the anti-plane boundary layer can be discarded in the first and third equalities of (12.10) with the formal asymptotic error (12.5). Moreover, it follows from (10.7), (10.4) and (12.3) that

$$s_{110}^{(r+q-p)} = s_{111}^{(r+q-p)} = 0 \quad (0 \leqslant r+q-p < q-2p+c)$$
$$s_{120}^{(r)} = 0 \quad (0 \leqslant r < q-p), \qquad s_{130}^{(r)} + \frac{1}{3} s_{132}^{(r)} = 0 \quad (0 \leqslant r < q-2p+c)$$
$$(\alpha_1 = 0)$$

Hence, the equalities (12.6) and (12.8) are obtained.

There results from (12.1) that near the free edge the anti-plane layer P_a generally possesses (for $p \neq 0$) a greater intensity, in the asymptotic sense, than the plane boundary layer P_b. We can write conditionally that

$$P_b = O(\varkappa^{-p+c}) P_a$$

(this means that the ratio of the greatest stresses or the greatest displacements in P_b and P_a is a quantity of the order of $O(\varkappa^{-p+c})$).

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*) Here and always when speaking of compliance with inequalities, the constraint (10.5) is taken into account.

It hence follows that in the total boundary layer P, which is the sum of P_a and P_b, the asymptotic error (12.5) admitted in the plane boundary layer will correspond to the formal asymptotic error (12.7).

13. Let the shell edge passing along the line $\alpha_1 = 0$ be rigidly clamped and the following boundary conditions of elasticity theory should be satisfied thereon

$$v_1^* = v_1 + hU_{1a} + hU_{1b} = 0, \quad v_2^* = v_2 + hV_{2a} + hV_{2b} = 0$$
$$v_3^* = v_3 + hU_{3a} + hU_{3b} = 0 \qquad (\alpha_1 = 0)$$

For this case we assume that

$$\lambda = p + d, \quad \mu = q + d \tag{13.1}$$

Then proceeding as in Sect. 12, and taking account of the first formula in (3.2), we obtain the following sequence of boundary conditions:

$$v_{10}^{(l)} + \zeta v_{11}^{(l)} + RU_{1a}^{(l-3q+3p)} + RU_{1b}^{(l-q+p)} = 0 \qquad (0 \leqslant l < 2q - 2p)$$
$$v_{20}^{(l)} + \zeta v_{21}^{(l)} + RV_{2a}^{(l-2q+2p)} + RV_{2b}^{(l-2q+2p)} = 0 \quad (0 \leqslant l < 2q - 2p) \tag{13.2}$$
$$v_{30}^{(l)} + \zeta v_{31}^{(l)} + \zeta^2 v_{32}^{(l)} + RU_{3a}^{(q-3q+2p+c)} + RU_{3b}^{(l-q+c)} = 0 \quad (0 \leqslant l < 2q - p - c)$$
$$(\alpha_1 = 0)$$

The customary constraints within which the linear law of variation of v_1, v_2 over ζ remains true, are taken for l in the first two equalities; the upper bound for l in the third equality is $2q - p - c$, which is greater than $2q - 2p$ for $p \neq 0$. In this connection it is considered that v_3 is defined by (10.10).

Let us require that the superscript l in the first and third equalities of (13.2) be constrained by the respective inequalities

$$0 \leqslant l < q - p, \quad 0 \leqslant l < q - c$$

Then, by virtue of (4.3), the boundary layer quantities drop out and the mentioned equalities become

$$v_{10}^{(l)} + \zeta v_{11}^{(l)} = 0 \qquad (0 \leqslant l < q - p)$$
$$v_{30}^{(l)} + \zeta v_{31}^{(l)} + \zeta^2 v_{32}^{(l)} = 0 \qquad (0 \leqslant l < q - c) \qquad (\alpha_1 = 0) \tag{13.3}$$

They can be discarded since it is clarified below that (13.3) is a consequence of some other relationships. We shall hence consider the values of l in the first and third equalities of (13.2) to be constrained by the respective inequalities

$$q - p \leqslant l < 2q - 2p, \qquad q - c \leqslant l < 2q - p - c$$

Using this, let us replace the superscript l as follows:

$$l - q + p = r, \quad l - q + c = r$$

The first and third equalities of (13.2) can hence be written

$$EU_{1b}^{(r)} = a_0 + a_1\zeta, \qquad EU_{3b}^{(r)} = b_0 + b_1\zeta + b_2\zeta^2 \quad (\alpha_1 = 0, \ 0 \leqslant r < q - p) \tag{13.4}$$

where

$$a_0 = -\frac{E}{R} v_{10}^{(r+q-p)}, \qquad a_1 = -\frac{E}{R} v_{11}^{(r+q-p)}$$
$$b_0 = -\frac{E}{R} v_{30}^{(r+q-c)}, \quad b_1 = -\frac{E}{R} v_{31}^{(r+q-c)}, \quad b_2 = -\frac{E}{R} v_{32}^{(r+q-c)} \tag{13.5}$$

Let us consider (13.4), (13.5) as endface conditions for the plane boundary layer, and let us require that the corresponding damping conditions be satisfied.

In this case they can be expressed by three equalities

$$v_{11}^{(r+q-p)} = f v_{32}^{(r+q-c)}, \quad v_{30}^{(r+q-c)} = g v_{32}^{(r+q-c)}, \quad v_{10}^{(r+q-p)} = m v_{31}^{(r+q-c)}$$
$$(\alpha_1 = 0, \ 0 \leqslant r < q - p) \tag{13.6}$$

in which the coefficients f, g, m are expressed by the formulas

$$m = -\frac{P_1^{[w1]}}{P_1^{[u0]}}$$
$$f = \frac{P_2^{[w0]} P_3^{[w2]} - P_2^{[w2]} P_3^{[w0]}}{P_2^{[u1]} P_3^{[w0]} - P_2^{[w0]} P_3^{[u1]}} \tag{13.7}$$
$$g = \frac{P_2^{[w2]} P_3^{[u1]} - P_2^{[u1]} P_3^{[w2]}}{P_2^{[u1]} P_3^{[w0]} - P_2^{[w0]} P_3^{[u1]}}$$

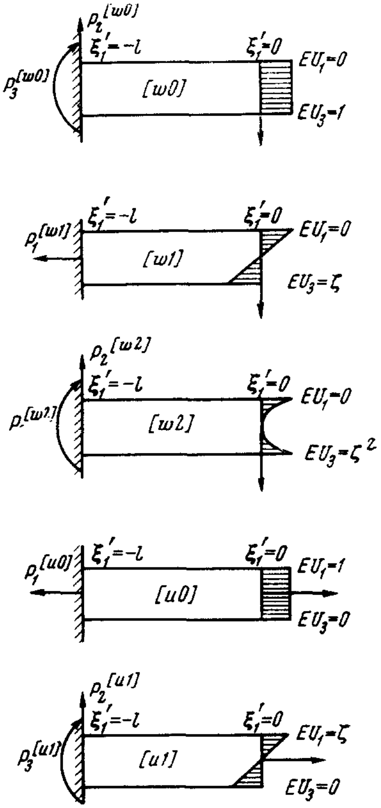

Fig. 2

and the quantities on the right sides of these equalities can be obtained as a result of solving the five plane problems pictured in Fig. 2 (nonzero displacement patterns are shown on the endface $\xi_1' = 0$ and nonzero reactions in the formulas under discussion – on the endface $\xi_1' = -l$).

To prove the equalities (13.6), (13.7), let us turn to the plane problem considered in Sect. 8. The endface conditions (8.3) represent a generalization of the equalities (13.4), and the kinematic damping conditions (8.4) can be utilized. In the case under discussion, it is necessary to consider that $P_\rho^* \equiv 0$ in (8.4) since the superscript r in (13.4) does not exceed q, and therefore, the plane boundary layer equations are homogeneous (see Sect. 5). In addition, only terms containing a_0, a_1, b_0, b_1, b_2 are retained in the sums in (8.4), and it is seen from Fig. 2 that the P_1 differ from zero only in the state of stress $S^{[u0]}$ and $S^{[w1]}$, and the reactive transverse force and moment P_2, P_3 are nonzero only in the states of stress $S^{[u1]}$, $S^{[w0]}$ and $S^{[w2]}$. Hence, in the case under consideration, the kinematic damping conditions become

$$P_1 \equiv a_0 P_1^{[u0]} + b_1 P_1^{[w1]} = 0, \qquad P_2 \equiv a_1 P_2^{[u1]} + b_0 P_2^{[w0]} + b_2 P_2^{[w2]} = 0$$
$$P_3 \equiv a_1 P_3^{[u1]} + b_0 P_3^{[w0]} + b_2 P_3^{[w2]} = 0$$

Hence, taking account of (13.5), we obtain (13.6), (13.7).

We consider the three kinematic damping conditions (13.6), (13.7) as boundary conditions for the interior state of stress, and append the second equality in (13.2) thereto as a fourth condition. Then, after having been converted to the form (13.4) and (13.5), the first and third equalities of (13.2) form endface conditions for the plane boundary layer.

Within the span of the formal asymptotic accuracy (4.4) taken here, the boundary conditions described above for the interior state of stress result in

$$v_{10}^{(r)} + \frac{\nu R}{E} m\left(s_{110}^{(r-q+p)} + s_{220}^{(r-q+p)}\right) = 0 \qquad (0 \leqslant r < 2q - p - c)$$
$$v_{20}^{(r)} = v_{30}^{(r)} = 0 \qquad (0 \leqslant r < 2q - 2p) \tag{13.8}$$
$$v_{11}^{(r)} + \frac{\nu R}{2E} f\left(s_{111}^{(r-q+p)} + s_{221}^{(r-q+p)}\right) = 0 \qquad (0 \leqslant r < 2q - 2p)$$
$$(\alpha_1 = 0)$$

and the endface conditions for a plane boundary layer with a formal asymptotic error (12.7) can be written as

$$EU_{1b}^{(r)} = \nu m\left(s_{110}^{(r)} + s_{220}^{(r)}\right) + {}^1/_2\nu f/\zeta\left(s_{111}^{(r)} + s_{221}^{(r)}\right) \tag{13.9}$$
$$EU_{3b}^{(r)} = {}^1/_2\nu g\left(g + \zeta^2\right)\left(s_{111}^{(r)} + s_{221}^{(r)}\right) + \nu\zeta\left(s_{110}^{(r)} + s_{220}^{(r)}\right)$$
$$(\alpha_1 = 0,\ 0 \leqslant r < q - p)$$

Let us prove the validity of the equalities (13.8) and (13.9). The quantities on the right sides of (13.6) can be expressed by using (10.12) and (10.13). Let us make the appropriate changes in superscripts in these equalities, and let us note that for $r < q - p$ the following inequalities are valid:

$$r + q - c < 3q - 2p, \quad r - q < 0, \quad r - q + c < 0$$

Discarding quantities with superscripts known to be negative, we obtain

$$v_{32}^{(r+q-c)} = -\frac{\nu R}{2E}\left(s_{111}^{(r)} + s_{221}^{(r)}\right) \qquad (0 \leqslant r < q - p) \tag{13.10}$$
$$v_{31}^{(r+q-c)} = -\frac{\nu R}{E}\left(s_{110}^{(r)} + s_{220}^{(r)}\right) \qquad (0 \leqslant r < q - c) \qquad (\alpha_1 = 0)$$

By virtue of (10.7) we have

$$s_{111}^{(r)} = s_{221}^{(r)} = 0 \qquad (r < q - 2p + c)$$

Hence

$$v_{32}^{(l)} = 0 \quad (0 \leqslant l < 2q - 2p), \qquad v_{31}^{(l)} = 0 \quad (0 \leqslant l < q - c) \qquad (\alpha_1 = 0) \tag{13.11}$$

There results from (13.11) and (13.6) that

$$v_{10}^{(l)} = 0 \quad (0 \leqslant l < q - p), \qquad v_{11}^{(l)} = 0 \quad (0 \leqslant l < 2q - 3p + c) \tag{13.12}$$
$$v_{30}^{(l)} = 0 \qquad (0 \leqslant l < 2q - 2p) \tag{13.13}$$
$$(\alpha_1 = 0)$$

Meanwhile, the inequalities

$$2q - 3p + c \geqslant q - p, \qquad 2q - 2p \geqslant q - c$$

hold, from which it follows that (13.3) could actually be ignored since these equalities are a consequence of (13.6) and (13.11). Substituting (13.10) into (13.6) we have

$$v_{11}^{(r+q-p)} = -(\nu R / 2E) f\left(s_{111}^{(r)} + s_{221}^{(r)}\right) \qquad (0 \leqslant r < q - p) \tag{13.14}$$
$$v_{30}^{(r+q-c)} = -(\nu R / 2E) g\left(s_{111}^{(r)} + s_{221}^{(r)}\right)$$
$$v_{10}^{(r+q-p)} = -(\nu R / E) m\left(s_{110}^{(r)} + s_{220}^{(r)}\right) \qquad (0 \leqslant r < q - c)$$
$$(\alpha_1 = 0)$$

Turning to the boundary conditions (13.2), let us note that the second of these equalities can be replaced by $v_{20}^{(l)} = 0 \qquad (\alpha_1 = 0,\ 0 \leqslant l < 2q - 2p)$ (13.15)

since the quantities associated with the boundary layers are known to have negative superscripts therein. Furthermore, it follows from (13.11) and (13.15) that $v_{21}^{(l)}$ equals zero on the boundary for $l < 2q - 2p$. Indeed, any expression comprised of $v_{20}^{(l)}$ and $v_{30}^{(l)}$ or of their derivatives with respect to α_2 should vanish because of (13.11) and (13.15) if $l < 2q - 2p$, and (10.8) shows that the quantity $v_{21}^{(l)}$ has precisely that form.

It has thus been shown that the boundary conditions (13.2) are equivalent to (13.4), (13.5) and (13.15) and the kinematic damping conditions reduce to the equalities (13.11), (13.12), (13.13).

The equalities (13.11) can be discarded: by virtue of (10.12), (10.13) and (10.7) they are satisfied not only on the boundary but everywhere. The equalities (13.12), (13.13) and (13.15) are reduced to the form (13.8) by using (13.14), and the equalities (13.4) and (13.5) are converted into the endface conditions (13.9) by using (13.10) and (13.13). The statement required is proved.

The endface condition for the anti-plane boundary layer does not appear among the relationships deduced. However, it is seen from (13.1) that in this case, in the notation utilized in Sect. 12

$$P_a = O(\varkappa^{-q+p})\, P_b$$

and this means that if the plane boundary layer has been constructed with the formal asymptotic error (12.7), then the anti-plane boundary layer should not be taken into account within the span of such accuracy.

14. Let the following mixed boundary conditions of elasticity theory be satisfied on the edge passing along the line $\alpha_1 = 0$:

$$s_{11}^* \equiv s_{11} + S_{11a} + S_{11b} = 0, \qquad v_2^* = v_2 + hV_{2a} + hV_{2b} = 0 \qquad (14.1)$$
$$v_3^* \equiv v_3 + hU_{3a} + hU_{3b} = 0$$

(we consider that they model a hinge-supported edge). In this case, assuming

$$\lambda = p + d, \quad \mu = q + d \qquad (14.2)$$

we obtain the sequence of boundary conditions

$$s_{110}{}^{(l)} + \zeta s_{111}{}^{(l)} + S_{11a}{}^{(l-2q+2p)} + S_{11b}{}^{(l)} = 0 \qquad (0 \leqslant l < 2q - 2p)$$
$$v_{20}{}^{(l)} + \zeta v_{21}{}^{(l)} + RV_{2a}{}^{(l-2q+2p)} + RV_{2b}^{(l-2q+2p)} = 0 \qquad (0 \leqslant l < 2q - 2p) \qquad (14.3)$$
$$v_{30}{}^{(l)} + \zeta v_{31}{}^{(l)} + \zeta^2 v_{32}{}^{(l)} + RU_{3a}{}^{(l-3q+2p+c)} + RU_{3b}{}^{(l-q+c)} = 0 \qquad (0 \leqslant l < 2q - p - c)$$
$$(\alpha_1 = 0)$$

to which two static and one kinematic damping conditions must be appended.

The first and fourth equalities in (6.4) are the static damping conditions in the considered case. They can be transformed by the scheme described in Sect. 12. From the first boundary condition of (14.3), we obtain, exactly as in Sect. 12, the first and fourth equalities of (12.4) in which the superscript $l - 2q + 3p - c$ must be replaced by $l - 2q + 2p$ for the quantities associated with the anti-plane boundary layer, and the superscript $l - q + p$ – by l for the quantities associated with the plane boundary layer.

Hence, by using (7.1) and (7.2) after quantities with superscripts known to be negative have been discarded, we obtain the required static damping conditions

$$2s_{110}{}^{(l)}|_{\alpha_1=0} + \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} Rk_{20} S_{22b}{}^{(l-q)} A_{10}\, d\xi_1 = 0 \qquad (0 \leqslant l < 2q - 2p) \qquad (14.4)$$

(cont.)

$$\frac{2}{3} s_{111}{}^{(l)}|_{\alpha_1=0} + \int_{-1}^{+1} d\zeta \int_{\infty}^{0} \left[Rk_{20}\zeta S_{22b}{}^{(l-q)} + \frac{R}{R_{20}} A_{10}\xi_1 S_{22b}{}^{(l-q)}\right] A_{10} d\xi_1 = 0$$

from which there follows, in particular, that

$$s_{110}{}^{(l)} = s_{111}{}^{(l)} = 0 \qquad (\alpha_1 = 0,\ 0 \leqslant l < q) \tag{14.5}$$

Let us constrain the values of l in the third condition in (14.3) by the inequalities $0 \leqslant l < q - c$, and let us discard members referring to the boundary layers by virtue of (4.3). The equality

$$v_{30}{}^{(l)} + \zeta v_{31}{}^{(l)} + \zeta^2 v_{32}{}^{(l)} = 0 \qquad (\alpha_1 = 0,\ 0 \leqslant l < q - c) \tag{14.6}$$

is then obtained which, as will be elucidated below, cannot be taken into account since it is a consequence of other relationships. Therefore l in the third equality of (14.3) can be bounded by the inequalities

$$q - c \leqslant l < 2q - p - c$$

and, on this basis the superscripts in the first and third equalities of (14.3) can be replaced by $l = r$ and $l - q + c = r$ respectively. Then, taking (4.3) into account, these equalities can be written as

$$S_{11b}{}^{(r)}|_{\alpha_1=0} = f(\zeta) \quad (0 \leqslant r < 2q - 2p), \qquad EU_{3b}{}^{(r)}|_{\alpha_1=0} = c_0 + c_1\zeta + c_2\zeta^2$$
$$(0 \leqslant r < q - p) \tag{14.7}$$

where

$$f(\zeta) = -s_{110}{}^{(r)} - \zeta s_{111}{}^{(r)}$$
$$c_0 = -\frac{E}{R} v_{30}{}^{(r+q-c)}, \quad c_1 = -\frac{E}{R} v_{31}{}^{(r+q-c)}, \quad c_2 = -\frac{E}{R} v_{32}{}^{(r+q-c)} \tag{14.8}$$

They are the endface conditions for the plane boundary layer, and the corresponding kinematic damping condition is

$$v_{30}{}^{(r)} = 0 \qquad (\alpha_1 = 0,\ 0 \leqslant r < 2q - 2p) \tag{14.9}$$

In order to show this, let us note that the plane problem with endface conditions of the form (14.7) has been considered in Sect. 9. The equality (9.2) is its single kinematic damping condition. Let us apply it to the case under discussion, let us consider the superscript r in the first equality of (14.7) to be bounded by the same inequalities as in the second. Then r will be sufficiently small, so that Eqs. (5.1) of the plane boundary layer would be homogeneous, and (14.5) would be satisfied, whereupon $f(\zeta) = 0$. Moreover, in our case the sum in (9.1) consists of the first three members, and it can be considered that only $S^{(w0)}, S^{(w1)}, S^{(w2)}$ (Fig. 3) among the states of stress introduced in Sect. 9 are not zero, where it is evident that $S^{(w1)}$ yields a zero transverse reactive force. Therefore (9.2) becomes

$$c_0 P_2{}^{(w0)} + c_2 P_2{}^{(w2)} = 0$$

from which we obtain, by taking account of (14.8) and (13.10)

$$v_{30}{}^{(r+q-c)} = e\frac{\nu R}{2E}\left(s_{111}^{(r)} + s_{221}^{(r)}\right) \qquad (\alpha_1 = 0,\ 0 \leqslant r < q - p) \tag{14.10}$$

in which

$$e = P_2{}^{(w2)} / P_2{}^{(w0)} \tag{14.11}$$

Here $P_2^{(w0)}$ and $P_2^{(w2)}$ are transverse reactions in the first and third problems pictured in Fig. 3.

The quantity on the right side of (14.10) is not zero only for $r \geqslant q - 2p + c$ by virtue of (10.7). Hence, the discussed equality (14.9) results.

Let us note that (14.9) and (13.10) hold for $v_{30}^{(l)}, v_{31}^{(l)}, v_{32}^{(l)}$. It follows therefrom that within the range of variation of l shown for (14.6), all the quantities listed vanish, and this means that (14.6) is satisfied automatically.

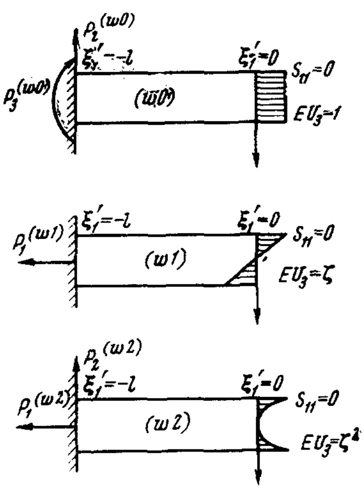

Fig. 3

The second equality in (14.3) is identical to the second equality in (13.2) and also reduces to the form (13.15).

Thus, four boundary conditions expressed by the two equalities (14.4) and (13.15), (14.9), are obtained for the interior state of stress on the hinged edge, and they permit construction of the interior state of stress with the formal asymptotic error (4.4). Endface conditions (14.7), (14.8) which permit its construction with the formal asymptotic error (12.7) are obtained for the plane boundary layer. In the first equality (14.7) it is sufficient to constrain the value of r by the inequality $r < q - p$, as it was done in deriving the kinematic damping condition.

By using (13.8), (14.5) and (14.10) the endface conditions (14.7), (14.8) can be converted to the form

$$S_{11b}^{(r)} = 0$$
$$EU_{3b}^{(r)} = {}^1/_2\nu(-e + \zeta^2)(s_{111}^{(r)} + s_{221}^{(r)}) + \nu\zeta(s_{110}^{(r)} + s_{220}^{(r)})$$
$$(\alpha_1 = 0,\ 0 \leqslant r < q - p) \tag{14.12}$$

There is no endface condition for an anti-plane boundary layer in the hinge-supported edge case. It also need not be constructed within the accuracy (12.7) as follows from (14.2).

15. The boundary conditions in the preceding sections have been formulated relative to the coefficients of the expansions (10.2), (10.3) and (11.3). Multiplying the relationships obtained by powers of $\varkappa$ selected in a suitable way, and adding, we can return from the expansions to the initial quantities. It is hence necessary to take account of the form of the mentioned expansions, (12.1), (13.1) and (14.2) for the numbers λ, μ and the first equality in (3.2). Moreover, it is necessary to disclose the meaning of d_{20} by formulas (3.4), (3.1) and to utilize the following, easily verifiable, summation rule:

$$\varkappa^a\left[\sum_{s=0}^{\sigma-1} \varkappa^{-s} P^{(s)} + O(\varkappa^{-\sigma})\right] = P$$

then

$$\varkappa^a \sum_{s=0}^{\sigma-1} \varkappa^{-s} P^{(s-b)} = \varkappa^{-b} P + O(\varkappa^{a-\sigma})$$

The results of these computations are as follows.

Free edge. Boundary conditions for the interior state of stress are obtained from (12.3), (12.6) and (12.8)

$$2s_{110}|_{\alpha_1=0} = 0, \qquad 2s_{120}|_{\alpha_1=0} = 0 \tag{15.1}$$

$$2\left(s_{130} + \frac{1}{3}s_{132}\right)_{\alpha_1=0} - h\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \left[\frac{1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2} + k_{10}T_{23a}\right] A_{10} d\xi_1 + $$
$$+ \frac{h}{R_{20}}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} S_{22b}A_{10}d\xi_1$$

$$\frac{2}{3}s_{111}|_{\alpha_1=0} - h\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} \left[\frac{\zeta}{A_{20}}\frac{\partial T_{21a}}{\partial \alpha_2} - \frac{A_{10}\xi_1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2}\right] A_{10} d\xi_1 = 0$$

as are also the endface conditions for the plane and anti-plane boundary layers

$$S_{11b} = 0, \qquad S_{13b} = (3\zeta^2 - 1)\,s_{130} - \zeta s_{131} \qquad (\alpha_1 = 0) \tag{15.2}$$

$$T_{12a} = -\zeta s_{121} \qquad (\alpha_1 = 0) \tag{15.3}$$

R i g i d l y c l a m p e d e d g e. From (13.8), (13.9) we obtain the boundary conditions for the interior state of stress

$$v_{10} + (\nu h/E)\, m\,(s_{110} + s_{220}) = 0, \qquad v_{20} = 0$$
$$v_{30} = 0, \quad v_{11} + (\nu h/2E)\, f\,(s_{111} + s_{221}) = 0$$
$$(\alpha_1 = 0) \tag{15.4}$$

and the endface conditions for the plane boundary layer

$$EU_{1b} = \nu m(s_{110} + s_{220}) + {}^1\!/_2\ \nu f\zeta\,(s_{111} + s_{221})$$
$$EU_{3b} = \nu\zeta(s_{110} + s_{220}) - {}^1\!/_2\nu(\,g + \zeta^2)(s_{111} + s_{221}) \tag{15.5}$$
$$(\alpha_1 = 0)$$

Let us note that the quantity v_{11} in the fourth condition in (15.4) can be expressed in terms of v_{30} and v_{10} by using (10.8). Summing here, we obtain

$$v_{11} = -h\left(\frac{1}{A_1}\frac{\partial v_{30}}{\partial \alpha_1} + \frac{v_{10}}{R_1}\right) \tag{15.6}$$

H i n g e d e d g e. From (14.4), (14.9), (13.15) result the boundary conditions for the interior state of stress

$$2s_{110}|_{\alpha_1=0} + hk_{20}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} S_{22b}A_{10}d\xi_1 = 0, \quad v_{20}|_{\alpha_1=0} = 0, \quad v_{30}|_{\alpha_1=0} = 0 \tag{15.7}$$

$$\frac{2}{3}s_{111}|_{\alpha_1=0} + hk_{20}\int_{-1}^{+1} \zeta d\zeta \int_{-\infty}^{0} S_{22b}A_{10}d\xi_1 - \frac{h}{R_{20}}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10}\xi_1 S_{22b}A_{10}d\xi_1 = 0$$

and the endface conditions for the plane boundary layer

$$S_{11b} = 0, \;\; EU_{3b} = \nu\zeta(s_{110} + s_{220}) + \frac{\nu}{2}(-e + g)(s_{111} + s_{221}) \;\; (\alpha_1 = 0) \tag{15.8}$$

16. Let us examine the structure of the boundary conditions obtained in Sect. 15. The interior state of stress and boundary layer are not separated there. However, in the boundary conditions imposed on the interior state of stress the boundary layer is of secondary value in the asymptotic sense (with the single exception which is discussed below). This assertion is easily verified since estimates for the interior state of stress are obtained from (10.2), (10.3), (10.7) and (10.8), and estimates for the boundary layers – from the expansion (11.3) and the formulas (12.1), (13.1) and (14.2) for the numbers λ, μ, and the symbol $d(\cdot)\,/\,d\alpha_2$ can be estimated by using (3.3). Performing such an analysis, we see that only the third equality in (15.1), in which the first two members are in the following asymptotic relation

$$h\int_{-1}^{+1} d\zeta \int_{-\infty}^{0}\left[\frac{1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2}+k_{10}T_{23a}\right]A_{10}d\xi_1 = 0(\varkappa^{-q+2p-c})\left(s_{130}+\frac{1}{3}s_{132}\right)_{\alpha_1=0} \quad (16.1)$$

is the above-mentioned exception; this means that for $2p \geqslant q$, i.e. for $t \geqslant 1/2$ the anti-plane boundary layer and the interior state of stress in the third equality of (15.1) are commensurate.

Within the accuracy (4.3) the left side of (16.1) can be expressed

$$h\int_{-1}^{+1} d\zeta \int_{-\infty}^{0}\left[\frac{1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2}+k_{10}T_{23a}\right]A_{10}d\xi_1 = -\frac{2}{3}\frac{h}{A_{20}}\frac{\partial s_{121}}{\partial \alpha_2} \quad (16.2)$$

Indeed, the meaning of k_{10} is defined by (2.3), consequently

$$\frac{1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2}+k_{10}T_{23a}=\frac{1}{A_{10}A_{20}}\frac{\partial}{\partial \alpha_2}(A_{10}T_{23a}) \quad (16.3)$$

and therefore

$$h\int_{-1}^{+1} d\zeta \int_{-\infty}^{0}\left[\frac{1}{A_{20}}\frac{\partial T_{23a}}{\partial \alpha_2}+k_{10}T_{23a}\right]A_{10}d\xi_1 = \frac{h}{A_{20}}\frac{\partial}{\partial \alpha_2}\left\{\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} T_{23a}A_{10}d\xi_1\right\} \quad (16.4)$$

Let us integrate by parts the expression in the braces with respect to ζ, and let us take account of condition (6.2). We obtain

$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} T_{23a}A_{10}d\xi_1 = -\int_{-1}^{+1} \zeta d\zeta \int_{-\infty}^{0} \frac{\partial T_{23a}}{\partial \zeta}A_{10}d\xi_1$$

Meanwhile, it follows from (16.1) that the required accuracy (4.4) will be retained if the integral under discussion is evaluated with a formal asymptotic error of the order of $O(\varkappa^{-q+c})$, and since $c \geqslant 0$ it can be considered that T_a satisfies the homogeneous equations (5.2). Therefore

$$\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} T_{23a}A_{10}\,d\xi_1 = \int_{-1}^{+1} \zeta d\zeta \int_{-\infty}^{0} \frac{1}{A_{10}}\frac{\partial T_{12a}}{\partial \xi_1}A_{10}d\xi_1 = \int_{-1}^{+1} \zeta T_{12a}|_{x_1=0}d\zeta$$

The integrand on the right side can here be replaced by the boundary value s_{121} by using (15.3), from which formula (16.2) is obtained.

Now, an iteration process satisfying the boundary conditions of the preceding section can be formulated in which the interior state of stress, and the plane and anti-plane boundary layers are separated from each other in a known sense.

In the boundary conditions of the interior state of stress, let us discard those components which are generated by the boundary layer and are of secondary value, and let us express the component of (16.1) not subject to this rule by using (16.2). Then, within some accuracy the construction of the interior state of stress is isolated into an independent problem. Having solved it, we can turn to the construction of the boundary layers by considering terms generated by the interior state of stress to be known in the appropriate conditions. This results in the plane and anti-plane boundary layers also being separated from each other, and obvious iterations may henceforth be utilized. This is indeed the iteration process mentioned in Sect. 11, and the fact that it can be obtained successfully shows that consistent values of the indices λ, μ are actually defined by (12.1), (13.1) and (14.2).

17. Let us now show that if the solutions of some auxiliary problems are considered known, then the interior state of stress can be separated with all the asymptotic accuracy (4.4) taken here.

The auxiliary problems will be marked with numbers in square brackets, and they consist of construction damped solutions of the homogeneous equations of the plane or anti-plane elasticity theory problems in the half-strip

$$-\infty < A_{10}\xi_1 \leqslant 0, \qquad -1 \leqslant \zeta \leqslant +1$$

on whose sides there should be no exterior forces. Let us present the endface conditions which distinguish the auxiliary problems [1], [2], ..., [7] (the problem [1] is anti-plane, and the rest are plane):

[1]	[2]	[3]	[4]	[5]	[6]	[7]
$T_{12}=\zeta$	$S_{11}=0$	$S_{11}=0$	$EU_1=m$	$EU_1=f\zeta$	$S_{11}=0$	$S_{11}=0$
—	$S_{13}=3\zeta^2-1$	$S_{13}=\zeta$	$EU_3=\zeta$	$EU_3=g+\zeta^2$	$EU_3=\zeta$	$EU_3=-e+\zeta^2$

The existence of damped solution in all the auxiliary problems is assured. Problem [1] is solved easily by using trigonometric series and the validity of this assertion is verified directly in it. The endface loadings are self-equilibrated in problems [2], [3], and damping follows from the Saint-Venant principle. It derives from the results in [10] in problem [6], and finally, the numbers m, f, g, e (Sects. 8, 9) were determined from the damping conditions in problems [4], [5], [7].

The endface conditions (15.2), (15.3), (15.5) and (15.8) can be represented as linear combinations of the endface conditions of the auxiliary problems. Hence, the plane and anti-plane boundary layers originating near the edges with the fixing conditions considered above, are also linear combinations of solutions of the auxiliary problems. Namely near the free edge

$$T = T^{[1]}[s_{121}]_{\alpha_1=0}, \qquad S = S^{[2]}[s_{130}]_{\alpha_1=0} + S^{[3]}[s_{131}]_{\alpha_1=0} \tag{17.1}$$

near the rigidly clamped edge

$$S = \nu S^{[4]}[s_{110}+s_{220}]_{\alpha_1=0} + {}^1/_2\nu S^{[5]}[s_{111}+s_{221}]_{\alpha_1=0} = 0 \tag{17.2}$$

near the hinged edge

$$S = \nu S^{[6]}[s_{110}+s_{220}]_{\alpha_1=0} + {}^1/_2\nu S^{[7]}[s_{111}+s_{222}]_{\alpha_1=0} \tag{17.3}$$

The purpose posed is indeed achieved by the formulas obtained since quantities associated with the boundary layer can now be eliminated in the boundary conditions for the interior state of stress.

For convenience of comparison, we formulate corresponding results in terms of classical shell theory. To do this, we write formulas for the stress resultants, moments and displacements by utilizing (10.1), (10.6), (3.1) and (3.2). In the notation of [11], they are

$$\begin{gathered} T_i = 2hs_{ii0}, \quad S_1 = 2hs_{120}, \quad G_i = -{}^2/_3 h^2 s_{ii1}, \quad H_1 = {}^2/_3 h^2 s_{121} \\ N_i = -2h(s_{i30} + {}^1/_3 s_{i32}) \qquad (i=1,2) \\ u = v_{10}, \quad v = v_{20}, \quad w = v_{30}, \quad v_{11} = h\gamma_1 \end{gathered} \tag{17.4}$$

(γ_1 is the elastic angle of rotation).

Let us express quantities referring to the boundary layer in the boundary conditions for the interior state of stress, i. e. in (15.1), (15.4), (15.6) and (15.7), by using (17.1), (17.2) or (17.3), and let us note that (16.2), as well as an equality of the form (16.3), hold, and let us transfer to the stress resultants, moments and displacements of shell theory by means of (17.4). We obtain

boundary conditions at the free edge

$$T_1 = 0, \qquad S_1 = 0$$

$$N_1 + \frac{1}{A_2}\frac{\partial H_1}{\partial \alpha_2} - \frac{h^2}{R_2}a^{[3]}s_{131} = 0, \quad G_1 + ha^{[1]}\frac{1}{A_2}\frac{\partial H_1}{\partial \alpha_2} = 0 \qquad (17.5)$$

boundary conditions at a rigidly clamped edge

$$2Ehu_1 + \nu hm(T_1 + T_2) = 0, \quad 2Ehu_2 = 0$$
$$2\ Ehw = 0, \quad 2Eh\gamma_1 - (\nu/h)\ f(G_1 + G_2) = 0 \qquad (17.6)$$

boundary conditions at a hinge supported edge

$$T_1 + \nu ha^{[6]}k_2(T_1 + T_2) = 0, \qquad v = 0 \qquad (17.7)$$
$$w = 0, \ G_1 - \nu ha^{[7]}k_2(G_1 + G_2) + \nu h^2\frac{b^{[6]}}{R_2}(T_1 + T_2) = 0$$

Here the superscripted letters a and b have the following meaning:

$$a^{[1]} = \frac{3}{2}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} (\zeta T_{12a}^{[1]} - A_{10}\xi_1 T_{23a}^{[1]})\, A_{10}d\xi_1$$

$$a^{[3]} = \int_{-1}^{+1} d\zeta \int_{-\infty}^{0} S_{22b}^{[3]} A_{10}d\xi_1, \quad a^{[6]} = \frac{1}{2}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} S_{22b}^{[6]} A_{10}d\xi_1 \qquad (17.8)$$

$$a^{[7]} = \frac{3}{4}\int_{-1}^{+1} \zeta d\zeta \int_{-\infty}^{0} S_{22b}^{[7]} A_{10}d\xi_1, \quad b^{[6]} = \frac{1}{2}\int_{-1}^{+1} d\zeta \int_{-\infty}^{0} A_{10}\xi_1 S_{22b}^{[6]} A_{10}d\xi_1$$

18. It has been shown in [1] that the accuracy of constructing the interior state of stress can be raised substantially by an insignificant modification of the classical shell theory equations. The modification is just a suitable selection of the elasticity relationships; they are presented in an arbitrary coordinate system in [1], and are written down in [12] for a shell referred to the lines of curvature.

The formal asymptotic error of the modified shell theory equations is of the order of (4.4). It has been shown that such accuracy is the ultimate for equations obtained within the scope of the customary representations of the classical shell theory, i. e. for equations constructed without introducing new concepts and without raising the order of the equations.

Modified boundary conditions, which have been obtained here in the form of the equalities (17.5)–(17.8), for the free, rigidly clamped and hinge-supported edges, correspond in accuracy to the modified equations.

In classical shell theory neither the equations nor the boundary conditions assure the accuracy (4.4). The equations of classical theory (for insufficiently accurately chosen elasticity relationships) lead to errors of the order of

$$\varepsilon = \max\{O(h_*),\ O(h_*^{2-2t})\} \qquad (18.1)$$

This has been deduced in [13, 14] and verified in [1].

Errors in the boundary conditions of classical theory are easily estimated by examining the additional terms which appear upon going over to the modified boundary conditions. As is mentioned in [2], they can exceed (4.4) by achieving the order

$$\varepsilon = O(h_*^{1-t})$$

Together, the modified equations and modified boundary conditions form a two-dimensional theory of the interior state of stress which permits the determination of the interior state of stress of a shell without going beyond the habitual concepts of classical theory with an ultimately possible accuracy on the order of (4.4). This result is not at all universal, it refers just to those interior states of stress whose asymptotics is determined by (10.2), but the majority of problems of practical importance possess this property (see [1]).

The results obtained here permit the determination of the boundary layer also, i.e. the investigation of edge stresses which are outside the range of classical shell theory. It is necessary to consider the construction of the boundary layer as the second stage in shell analysis. It is performed after the interior state of stress has been found, and consists in constructing a linear combination of the solutions of the seven auxiliary problems introduced in Sect. 17 by means of (17.1)–(17.3). The boundary values of the interior stresses obtained earlier are the coefficients of this linear combination.

The proposed theory of edge stresses is of the accuracy of the theory of the interior state of stress described above. Namely, a boundary layer is constructed with a formal asymptotic error of the order of (12.7).

In combination, the theory of the interior state of stress and the theory of the boundary layer can be considered as the initial approximation of some iteration process permitting a formal approach, as close as desired, to the solution of an appropriate boundary value problem of elasticity theory.

For an actual analysis of shells by the method proposed it is necessary to have the solution of the seven auxiliary problems (Sect. 17) and also to know the coefficients m, f, g, e in the corresponding kinematic damping conditions (Sects. 8, 9), and the coefficients a, b defined by (17.8). Let us note that for an isotropic shell the conditions of all the auxiliary problems, just as the requirements and formulas governing the values of a, b, m, f, g, e are independent of geometric properties of the shell, and almost independent of its physical properties (there exists only a slight dependence on Poisson's ratio ν). All these quantities are dimensionless, have the form $O(h_*^{\circ})$, and for anisotropic shells it is sufficient to evaluate them once as a function of the parameter ν. Just as the stresses determined by the auxiliary problems far from the corners of the half-strip, the numbers a, b not only remain finite as $h_* \to 0$, but are also not too different from unity. Meanwhile, m, f, g, e will be small in absolute value. This is evident from physical considerations; according to (13.7), (14.11), m, f, g, e are comprised of the reactive forces and moments pictured in Fig. 2, 3, but they should evidently be considerably smaller in problems $[w1]$, $[w2]$, $(w2)$ than in problems $[w0]$, $[u0]$, $[u1]$, $(w0)$, and it follows from the structure of the formulas (13.7), (14.11) that the absolute values of m, f, g, e will be small. This is also confirmed by calculations performed on the basis of results in [10]; for $\nu = 0.3$ we obtain

$$e = -0.20;\ f = 0.097;\ g = -0.25$$

The modified boundary conditions differ from the boundary conditions of classical theory by some number of corrective terms reflecting the influence of the boundary layer on the interior state of stress of the shell. As has been shown in [2], they are a generalization of those corrections which were introduced by Kirchhoff. These latter may be considered as first approximation corrections while the next approximations have already been taken into account in the modified boundary conditions (this result was obtained in [15] for a plate). This is seen clearly in Sect. 16. The term (16.1), corresponding to the Kirchhoff correction, turns out to be the single member associated with the boundary layers which is not known to be less than the members associated with the interior state of stress.

The coefficients in the modified boundary conditions are not identical in absolute value. The correction terms in the static boundary conditions enter in the coefficients a, b, which are commensurate with unity. They enter in the kinematic boundary conditions with the coefficients νm, νf, νg, νe, which are small as compared with one (although finite as $h_* \to 0$). This means that the errors in the kinematic boundary conditions in classical shell theory is less in practice, than the error in the static boundary conditions.

It certainly does not follow from the above that an increase in the stiffness of fixing the edge diminishes the relative role of the boundary layer stresses. From (17.1)–(17.3) it follows that the order of the edge stresses of the boundary layer is determined by the order of the coefficients in the solutions of the auxiliary problems, i.e. the edge stresses will generally be of the same order as the greatest interior stresses. Only a free edge of a shell might be an exception (but not of a plate), if the variability of the interior state of stress therein is not too great.

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ON THE EFFECT OF AN AXISYMMETRIC NORMAL LOADING ON AN ELASTIC SPHERE

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A solution of the problem of deformation of a sphere under normal loadings is obtained by quadratures. The Green's function of the boundary value problem is written out in finite form. In contrast to an analogous series solution [1], the solution found admits of nonsmooth loadings. As an example, the problem of compression of a sphere by concentrated forces is solved in closed form; the solution is expressed in terms of a hypergeometric function.

It is known from [1] that the solution of the equilibrium equations of an elastic body in displacements

$$\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u}-\operatorname{rot}\operatorname{rot}\mathbf{u}=0$$

with the boundary conditions

$$\tau_{r\theta}=0,\quad \sigma_r=\sigma(\theta),\quad \tau_{\varphi r}=0\quad \text{for } r=R$$

in a spherical coordinate system $r,\ \theta,\ \varphi$ has the following form:

$$u_r=\frac{R}{4\pi G}\int_0^\pi\sigma(\alpha)\sin\alpha\, d\alpha\int_0^{\frac{1}{2}\pi}d\psi\left\{\sum_{n=2}^\infty P_n(\lambda)\left[A_{1n}\left(\frac{r}{R}\right)^{n+1}+A_{2n}\left(\frac{r}{R}\right)^{n-1}\right]+\frac{2(1-2\nu)}{1+\nu}\frac{r}{R}\right\}$$

$$u_\theta=\frac{R}{4\pi G}\int_0^\pi\sigma(\alpha)\sin\alpha\, d\alpha\frac{\partial}{\partial\theta}\int_0^{\frac{1}{2}\pi}d\psi\sum_{n=2}^\infty P_n(\lambda)\left[A_{3n}\left(\frac{r}{R}\right)^{n+1}+A_{4n}\left(\frac{r}{R}\right)^{n-1}\right]\qquad(1)$$

$$\lambda=\cos(\theta+\alpha)+2\sin\theta\sin\alpha\sin^2\psi$$

Here $P_n(\lambda)$ are Legendre polynomials, and the coefficients A_{in} are rational fraction